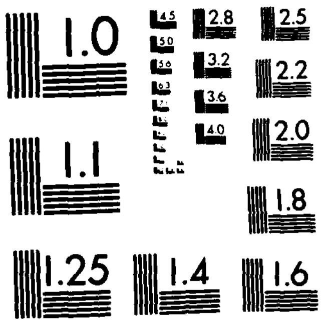
AD-A150 235 THE EFFECTS OF INTERNAL WAVES ON ACOUSTIC NORMAL MODES 1/2
(U) TEXAS UNIV AT AUSTIN APPLIED RESEARCH LABS
M C PENLAND DEC 84 ARL-TR-84-14 N00014-80-C-0490

UNCLASSIFIED F/G 20/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(1)

ARL-TR-84-14 Copy No. 12

THE EFFECTS OF INTERNAL WAVES ON ACOUSTIC NORMAL MODES

M. Cecile Penland

APPLIED RESEARCH LABORATORIES
THE UNIVERSITY OF TEXAS AT AUSTIN
POST OFFICE BOX 8029, AUSTIN, TEXAS 78713-8029

December 1984

Technical Report

Prepared for:

OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
ARLINGTON, VA 22217

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE EFFECTS OF INTERNAL WAVES ON ACOUSTIC NORMAL MODES		5. TYPE OF REPORT & PERIOD COVERED technical report
		6. PERFORMING ORG. REPORT NUMBER ARL-TR-84-14
7. AUTHOR(s) M. Cecile Penland		8. CONTRACT OR GRANT NUMBER(s) N00014-80-C-0490
9. PERFORMING ORGANIZATION NAME AND ADDRESS Applied Research Laboratories The University of Texas at Austin Austin, Texas 78713-8029		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Task 19
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Arlington, Virginia 22217		12. REPORT DATE December 1984
		13. NUMBER OF PAGES 146
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In describing the effects of internal waves on acoustic normal modes excited by a time-harmonic point source, all previous descriptions have neglected the existence of azimuthal fluctuations in the acoustic field. This study compares this azimuthally symmetric case with the case where azimuthal fluctuations in the acoustic field are taken into account. Although azimuthal variations enhance the randomization of interference effects so that different depth modes decorrelate more quickly in range, this enhancement is generally small. This

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. (cont'd)

mean energy flux vector <j> becomes radial at long ranges; the vertical components of <j> decay somewhat more quickly with range when azimuthal fluctuations are considered than when they are neglected.

TABLE OF CONTENTS

	Page
LIST OF TABLES	v
LIST OF FIGURES	v
CHAPTER 1	1
CHAPTER 2	9
CHAPTER 3	22
CHAPTER 4	41
CHAPTER 5	47
CHAPTER 6	65
CHAPTER 7	82
CHAPTER 8	90
APPENDIX A	99
APPENDIX B	103
APPENDIX C	107
APPENDIX D	111
APPENDIX E	119
APPENDIX F	122
APPENDIX G	125
APPENDIX H	130
REFERENCES	133

LIST OF TABLES

Table		Page
7.1	Calculation of a_{1n},	89
A.1	Coefficients used in Derivation of Munk Profile	100

LIST OF FIGURES

Figure		Page
1	Diagram for the Simultaneous Solution of the Simultaneous Equations (2.20) and (2.21)	19
2	Diagnostic Diagram for the General Case of an Ocean of Constant Depth and Constant N, Γ, c	20
3	The Väisälä (or Buoyancy) Frequency N(z)	24
4	The Wave Number Vector, Group Velocity and Hodograph of the Particle Velocity	30
5	Power Spectrum of Vertical Displacement of an Isotherm	36
6	Graph of Energy Density versus Frequency	37
7	Displacement Autospectrum versus Frequency	39
8	Contour for Calculation in Appendix H	132

Chapter 1

Everyone who has seen the ocean has watched the waves on its surface rise and fall. A gravitational restoring force acts on the vertically displaced water resulting in this near-harmonic motion. In the same manner, gravitational waves can be excited within the ocean's fluid volume wherever there is any change in mass density, either continuous or abrupt.

The spatial and temporal scales associated with this "internal" gravity wave motion are much different from those associated with particle motion caused by sound. Typical vertical displacements due to internal waves are 10-50 m while horizontal displacements are about 1 km and periods of oscillation run typically from about 20 minutes to 12 hours.[1] Particle displacements due to sound propagation in the ocean are dependent on the source strength, of course, but are generally on the order of Angstroms or microns while the period of a 100 Hz acoustic signal is five orders of magnitude smaller than the shortest internal wave periods! The effect of the internal wave motion is to move huge quantities of water very slowly, thus rearranging somewhat the sound speed field through which acoustic waves must travel.

What, then, is the cumulative effect of these sound speed fluctuations on the acoustic field? The quest for the answer to this question really began in 1972 when Garrett and Munk[2] introduced a model describing the statistical properties of the sound speed fluctuations due to internal waves. This model was improved and refined by subsequent theoretical and experimental work; for a summary of this work, the reader is referred to Vol. 80 (1975) of the Journal of Geophysical Research, which has been described by Briscoe[3] as a kind of birthday party to celebrate the field's growing up.

Armed with a statistical description of the medium, a group of scientists (the "JASON" group) employed Feynman path-integral techniques to describe propagation of acoustic rays through the internal wave field. A detailed exposition of this work has been published in monograph[4] and a summary has been presented by Flatté.[5]

The JASON theory is actually two theories; one is valid in the region of very weak sound speed fluctuations and "short" ($\sim 10^4$ m) ranges (the "unsaturated" region) and the other is valid in the region of stronger fluctuations or much longer ($\sim 10^6$ m) ranges (the "saturated" region). Consider acoustic propagation through a deterministic environment; sound travels from source to receiver along a set of eigenrays. Now turn on the fluctuations. In the unsaturated region, each of these equilibrium rays is still dominant although each of these rays and its associated ray tube is perturbed by the random sound speed inhomogeneities. In the saturated region, each ray has

been split into many "microrays" created by the fluctuations and separated by more than a vertical correlation length. The path-integral techniques describe acoustic propagation in each of these asymptotic regions; they are not expected to apply in the intermediate region where each ray has begun to split into microrays but these microrays are still correlated. Some corrections to the asymptotic theories have been made but the region of validity is not complete.

Experiment seems to validate most of the predictions of the JASON work in the regions where the theory is expected to apply. However, our understanding of the problem is still incomplete. As mentioned above, the behavior of the acoustic field at intermediate ranges is not totally understood. The path-integral techniques are invalid near caustics. Also, anisotropy and inhomogeneity in the sound speed fluctuations cause the Markov approximation used in the theory to break down at some lower limit on acoustic frequency. Although measures of this breakdown frequency (generally around 100 Hz) are available, the question remains as to what significant effect, if any, the internal waves can have on the acoustic field at very low acoustic frequencies.

The problem of low frequency ($\nu \lesssim 100$ Hz) acoustic propagation is particularly important since measurements of volume absorption in the oceanic waveguide typically involve precisely this type of transmission over ranges of many hundreds of kilometers.

The need for a low frequency theory of acoustic propagation through an internal wave field led some researchers[6-8] to con-

sider the effects of random sound speed inhomogeneities on acoustic normal modes. Certainly there are serious computational disadvantages to a normal mode description in the deep ocean; however, such a description is not only valid at low frequencies but also is free of the problems of caustics and the difficulties connected with the boundary between the saturated and unsaturated regions described above. In addition, normal modes offer a framework to describe energy transfer into the ocean bottom while the ray theory considered by the JASON work does not allow this bottom interaction. Hence, hoping to enhance the results already obtained through path-integral techniques, we turn to the stochastic differential equations describing acoustic normal mode propagation in a random environment.

The field of stochastic differential equations is a relatively new realm in mathematics; only 15 of the 81 references in Arnold's book,[9] Stochastic Differential Equations: Theory and Applications, were published before 1960. Therefore, it has often been necessary for physicists to knead their problems into somewhat different forms, with varying degrees of physical justifiability, in order to make progress using a recently developed mathematical arsenal. Specifically, the theory of stochastic partial differential equations is not well developed; true artistry is often necessary to reduce a physical problem to one dimension.

The problem of an acoustic wave propagation through an internal wave field is a prime example of an unsavory state of affairs. The fact that we may consider the sound speed field "frozen"

in time with respect to the acoustic field eliminates the temporal coordinate, but there are three spatial coordinates to contend with. The sound speed fluctuations are stationary in the horizontal plane but nonstationary in depth so that homogeneity cannot be invoked to reduce the dimensionality of the problems. Also, the fact that the horizontal correlation distance of the sound speed fluctuations is about 10 km while the acoustic wavelength is on the order of meters leads one to believe that the system may not be Markov!

In order to gain some knowledge of the system, Kohler and Papanicolaou[8] simplified the problem considerably by investigating acoustic normal mode propagation of a single tone through a cylindrically symmetric sound speed field. That is, the ocean sound speed knows where the sound source has been deployed in order to rise and fall symmetrically around it. The sound speed field was also assumed to be stationary in range. In this work, Kohler and Papanicolaou[8] laid the groundwork for the present study. They employed a theory which they introduced three years earlier[10] in order to derive formal propagation equations, although they did not consider a particular spectrum of sound speed fluctuations (if they had, they might have found some trouble arising from reversing a limit and an integral). The main contribution of their work was the introduction of rigorous scaling techniques by which it is possible to approximate the system by a Markov diffusion process (Appendix C) along with an estimate of the error based on the "smallness parameter" ε which characterizes the strength of the sound speed fluctuations.

About the same time, Dozier and Tappert[6-7] published two landmark papers which considered the effects of internal waves on time-harmonic acoustic normal modes. Using a nonrigorous perturbation technique, they were able to corroborate the formal coupled mode equations of Kohler and Papanicolaou. In addition, they found that the scattering coefficients, independent of range, represented contributions from the internal wave spectrum at a horizontal wave number equal to the difference of two acoustic normal mode wave numbers.

In the course of their work, Dozier and Tappert choose a direction of propagation and integrate the spectrum of the fluctuations over the wave number transverse to that direction. They then assume that this modified spectrum, with its corresponding sound speed fluctuations, can be used with a cylindrically symmetric stochastic Helmholtz equation to describe acoustic propagation. Although these manipulations reduce the system to a soluble one-dimensional problem, the attempt to localize a normal mode description is disturbing. In consequence of this method any transverse fluctuations in the acoustic field are eliminated not only on average but also for each realization, even though the sound speed fluctuations themselves vary randomly in azimuth.

Crucial to their derivation of the coupled mode equations is the assumption of "random phases". That is, it is assumed that correlations between different normal modes do not persist beyond a correlation length of the sound speed fluctuations. It will be shown

in the present study that when azimuthal fluctuations in the acoustic field are artificially prohibited but random phases are not assumed, significant correlation between different depth modes can persist for many ($\lesssim 25$) correlation lengths, but that the form of the coupled mode equations obtained by Dozier and Tappert is nevertheless recovered. In addition, we use the full two-dimensional spectrum of sound speed fluctuations with the result that the scattering coefficients represent contributions from the internal wave spectrum at a horizontal vector wave number having a component in the radial direction equal to the difference of two acoustic normal mode wave numbers. The average energy flux is found to be radial at long ranges.

Nevertheless, in spite of the various simplifications and assumptions, the Dozier-Tappert work is important in that it was the first to consider the effects of an internal wave field on acoustic normal modes and to offer the concept of resonance scattering in the context of acoustic propagation.

A major consideration of this dissertation is the effect of azimuthal fluctuations on the average energy flux. For each realization, the random depth mode amplitudes are functions of range and azimuth. Each mode amplitude is expressed in terms of a set of azimuthal modes in order to reduce the system to one dimension. Using the scaling techniques of Papanicolaou and Kohler,[10] the system is approximated by a Markov diffusion process and moments of the random mode amplitudes are calculated using the corresponding Fokker-Planck

equation. The random acoustic pressure is shown to be stationary in the horizontal plane for an azimuthally symmetric source. The coupled equations for the autocorrelation function of the mode amplitudes derived by suppressing azimuthal acoustic fluctuations are still valid as long as each range function is interpreted as a sum over all the azimuthal modes. The correlation between different modes, however, decreases with range somewhat more quickly than in the azimuthally symmetric case. Again, the average energy flux becomes radial at long ranges.

The order in which the material contained here is presented is as follows. Chapter 2 reviews the thermodynamics of the ocean and presents arguments for including all of the internal wave effects on acoustic propagation in the equation of state. Chapter 3 discusses models of the deterministic and random components for the sound speed and the assumptions contained therein. Chapter 4 reviews deterministic acoustic normal mode theory and discusses the acoustic Poynting vector. Chapter 5 investigates the effects of internal waves on an (artificially) cylindrically symmetric acoustic field. Many of the concepts which would be obscured in a more complex calculation are presented here. Chapter 6 considers acoustic propagation through a random internal wave field where now the effects of azimuthal fluctuations in the acoustic field are taken into account. Chapter 7 presents numerical calculation of scattering coefficients for comparison with the Dozier-Tappert results. Chapter 8 is a detailed summary of the results.

Chapter 2

Most discussions of acoustics, and underwater acoustics in particular, begin with the wave equation for the acoustic pressure. It is not clear that this is a legitimate procedure in the present study since we consider two effects, the acoustic pressure and the motion of the water due to gravity, or internal, waves, which may interact. Of course, they do interact; otherwise, this dissertation would have no point! However, if we can assume that the scales of particle motion caused by the sound differ greatly from those caused by internal waves, we may assume that internal waves affect sound propagation only through the equation of state. We therefore begin our discussion with the equations of motion for ocean water particles due to the combined sound and internal wave fields. Much of the following thermodynamic description follows Eckart.[11,12]

Seawater is here treated as a solution of a single compound in pure water. The internal energy ε as a function of specific volume v, entropy η, and salinity S can be used to calculate the pressure p, absolute temperature T, and chemical potential μ; i.e.,

$$p = -\frac{\partial\varepsilon}{\partial v}$$

$$T = \frac{\partial\varepsilon}{\partial\eta}$$

$$\mu = \frac{\partial\varepsilon}{\partial S} \qquad (2.1)$$

Small increments of these quantities are, then,

$$\begin{pmatrix} \delta p \\ \delta T \\ \delta \mu \end{pmatrix} = \begin{pmatrix} -X & Y & K \\ -Y & Z & L \\ -K & L & M \end{pmatrix} \begin{pmatrix} \delta v \\ \delta \eta \\ \delta S \end{pmatrix} \quad . \qquad (2.2)$$

Some elements of the square matrix in Eq. (2.2) are related to familiar response functions:

$$X = \rho^2 c^2$$

$$Y = \frac{\rho}{a}(\gamma - 1)$$

$$Z = T/C_{vS} \qquad , \qquad (2.3)$$

where

$\rho = 1/v$ = mass density,

$c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\eta S}$ defines the sound speed,

C_{vS} = specific heat at constant volume and salinity,

C_{pS} = specific heat at constant pressure and salinity,

$\gamma = C_{pS}/C_{vS}$, and

$a = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{pS}$ = coefficient of thermal expansion.

The other elements are related to the various heats of diffusion:

$$H_{pv} = -T\left(\frac{\partial\eta}{\partial S}\right)_{pv} = T\,\frac{K}{Y}$$

$$H_{Tv} = -T\left(\frac{\partial\eta}{\partial S}\right)_{Tv} = -\,\frac{L}{Z} = C_{vS}L \quad . \tag{2.4}$$

$$H_{\mu v} = -T\left(\frac{\partial\eta}{\partial S}\right)_{\mu v} = T\,\frac{M}{L}$$

It is convenient to introduce here the coefficient of saline contraction $b = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{Tp}$ and to note that

$$H_{Tp} = H_{Tv} + C_{vS}\left(\frac{b}{a}\right)(\gamma-1) \quad . \tag{2.5}$$

The hydrodynamic equations appropriate to an ocean are

$$\rho\,\frac{Du}{Dt} + \rho(\underline{f}\times\underline{u}) + \nabla p - \rho\underline{g} = \nabla\cdot\underline{\underline{\Lambda}} + \underline{F} \quad , \tag{2.6a}$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0 \quad , \tag{2.6b}$$

$$\rho T\,\frac{D\eta}{Dt} + \nabla\cdot\underline{h} = G \quad , \tag{2.6c}$$

$$\rho\,\frac{DS}{Dt} + \nabla\cdot\underline{s} = 0 \quad , \tag{2.6d}$$

where

$D/Dt = \partial/\partial t + \underline{u}\cdot\nabla$ = total time, or convective, derivative,

$\underline{u}$ = particle velocity,

$g = g\hat{k}$, the gravitational acceleration,

$\underline{f}$ = twice the earth's angular velocity of rotation,

$\underline{\underline{\Lambda}}$ = stress tensor of molecular viscosity (for an explicit definition see Refs. 12 and 13),

$\underline{F}$ = result of all other forces such as, for example, an oscillating piston,

G = the heat generated by irreversible processes,

$\underline{h}$ = heat flux, and

$\underline{s}$ = salinity flux.

Note that $\hat{k}$ points vertically downward.

If we assume that the particle motion is small, we may linearize these equations by writing each quantity as a sum of its time-dependent and its static values. Static values will be subscripted with zeroes; time-dependent quantities will be subscripted with ones. Therefore, $p=p_0+p_1$, $\underline{u}=\underline{u}_1$, $\underline{F}=\underline{F}_1$, $\rho=\rho_0+\rho_1$, etc. Equations (2.6) are ranked in successive approximations, the zero-order set being the case when no time-dependent quantities are considered, the first-order set where products of zero- and first-order quantities are considered but not products of first-order quantities, and so on. Relations between zero-order quantities may be exploited in the first-order equations, thus preserving the ranking of the approximation.

In order to obtain the zero-order relations, we neglect $\underline{\underline{\Lambda}}$ and all time derivatives. We assume that S does not vary with latitude; thus, all zero-order quantities depend only on depth. Thus

$$\nabla p_o = \rho_o \underline{g} \tag{2.7}$$

and

$$\frac{\partial p_o}{\partial z} = c_o^2 \frac{\partial \rho_o}{\partial z} + Y_o \frac{\partial \eta_o}{\partial z} + K_o \frac{\partial S_o}{\partial z} \quad . \tag{2.8}$$

Combining these, we define

$$N^2(z) \equiv g\left(\frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z} - \frac{g}{c_o^2}\right) = - \frac{g}{\rho_o c_o^2}\left(Y_o \frac{\partial \eta_o}{\partial z} + K_o \frac{\partial S_o}{\partial z}\right)$$

$$= - \left(\frac{\partial T_o}{\partial z}\right)_{\eta_o S_o} \left(\frac{\partial \eta_o}{\partial z} + \frac{H_{p_o v_o}}{T} \frac{\partial S_o}{\partial z}\right) \quad . \tag{2.9}$$

The quantity N(z), called the Brunt-Väisälä or buoyancy frequency, has an important physical meaning. Consider a small mass of fluid displaced adiabatically a small vertical distance from its zero-order position and then allowed to move freely. The fluid element will oscillate about its original position with frequency N, provided that N is real. If N^2 is negative, the fluid motion is unstable; a slight perturbation will cause wide departures from the fluid element's original position.[14-16]

It is very useful to consider the concept of a "potential gradient"; that is, the measured gradient minus what the gradient

would be in a static, isohaline, isentropic ocean. These gradients will be subscripted with P. For example,

$$\frac{1}{\rho_o}\left(\frac{\partial\rho_o}{\partial z}\right)_p = \frac{1}{\rho_o}\frac{\partial\rho_o}{\partial z} - \frac{g}{c_o^2} \quad . \tag{2.10}$$

Thus, it is the potential density gradient which determines the ocean's static stability, i.e.,

$$N^2(z) = \frac{g}{\rho_o}\left(\frac{\partial\rho_o}{\partial z}\right)_p \quad . \tag{2.11}$$

Similarly

$$\left(\frac{\partial T_o}{\partial z}\right)_p = \frac{\partial T_o}{\partial z} - \left(\frac{\partial T_o}{\partial z}\right)_{\eta_o S_o} = \frac{\partial T_o}{\partial z} - \frac{g}{a_o c_o^2}(\gamma-1) \quad . \tag{2.12}$$

The first-order linearized hydrodynamic equations are

$$\rho_o \frac{\partial \underline{u}_1}{\partial t} + \nabla p_1 + \rho_o\, \underline{f}\times\underline{u}_1 = \nabla\cdot\underline{\underline{\Lambda}} + \underline{F} + \rho_1 \underline{g} \quad , \tag{2.13a}$$

$$\frac{\partial\rho_1}{\partial t} + \nabla\cdot(\rho_o \underline{u}_1) = 0 \quad , \tag{2.13b}$$

$$\frac{\partial\eta_1}{\partial t} + \underline{u}_1\cdot\nabla\eta_o = \frac{1}{\rho_o T_o}(G-\nabla\cdot\underline{h}) \quad , \tag{2.13c}$$

$$\frac{\partial S_1}{\partial t} + \underline{u}_1\cdot\nabla S_o = -\frac{1}{\rho_o}\nabla\cdot\underline{s} \quad , \tag{2.13d}$$

with appropriate boundary conditions. In many cases, it is more convenient to employ the following form of Eqs. (2.13):

$$\frac{\partial \underline{U}}{\partial t} + c_o \left[\nabla + \Gamma\hat{k}\right] P + N^2(z)\hat{k}W + \underline{f}\times\underline{U} = \underline{\mathcal{F}} \quad , \tag{2.14a}$$

$$\frac{\partial P}{\partial t} + c_o \left[\nabla - \Gamma\hat{k}\right] \cdot \underline{U} = \mathcal{G} \quad , \tag{2.14b}$$

$$\frac{\partial W}{\partial t} - \hat{k}\cdot\underline{U} = \frac{g}{c_o N^2(z)} \mathcal{G} \quad , \tag{2.14c}$$

where

$$\underline{U} = \underline{u}_1(\rho_o c_o)^{1/2} \quad , \qquad P = p_1(\rho_o c_o)^{-1/2} \quad ,$$

$$N^2(z)W = (\rho_o c_o)^{1/2} \left(\frac{\partial T_o}{\partial z}\right)_{\eta_o S_o} \left(\eta_1 + \frac{H_{p_o v_o}}{T_o} S_1\right) \quad ,$$

$$\underline{\mathcal{F}} = \left(\frac{c_o}{\rho_o}\right)^{1/2} (\nabla\cdot\underline{\underline{\Lambda}} + \underline{F}) \quad ,$$

$$\Gamma = \frac{1}{2} (\rho_o c_o)^{-1} \frac{\partial}{\partial z} (\rho_o c_o) - \frac{g}{c_o^2} \quad ,$$

$$\mathcal{G} = \frac{c_o}{g T_o} (\rho_o c_o)^{-1/2} \left(\frac{\partial T_o}{\partial z}\right)_{\eta_o S_o} \left[G - \nabla\cdot\underline{h} - H_{p_o v_o} \nabla\cdot\underline{s}\right] .$$

Boundary conditions are now

$$\hat{n}\cdot\underline{U} = 0 \tag{2.15a}$$

on rigid surfaces and

$$\frac{\partial P}{\partial t} = \frac{g}{c}\,\hat{k}\cdot\underline{U} \tag{2.15b}$$

at the surface of the ocean. We see that when $\mathcal{G}=0$, $W(\rho_0 c_0)^{-1/2}$ is simply the vertical displacement of the fluid particle.

If we consider a region of the ocean far from heat and pressure sources, we may take $\underline{\mathcal{F}}$ and $\mathcal{G}$ in Eqs. (2.14-2.15) to be zero. Then, the solutions of these equations may be time harmonic; time derivatives are replaced by the factor $-i\omega$, where ω is the angular frequency. Actually, a general limited solution to these equations as they stand will be a superposition of these time harmonic solutions of the free equations.

It should be remembered that P, W, and $\underline{U}$ are first-order quantities; Γ and N are zero-order quantities. Therefore, to first order, it seems that the hydrodynamic equations should decouple into a set describing only the internal waves and another set describing the acoustic propagation.

To support this assertion, we consider an instructively simple situation where S, c_0, and N are constant everywhere and Γ is zero. This situation is physically impossible for the ocean but not

so bad for the atmosphere (which, of course, doesn't have salinity). Our source terms are also taken to be zero. We do not neglect rotation, but we do specialize it by making our system a flat layer of thickness H rotating about a vertical axis with angular velocity $\underline{f}=-f\hat{k}$. The solutions to the field Eqs. (2.14) are

$$P = FP_1(z)\, e^{-i\omega t} \quad , \tag{2.16a}$$

$$W = FW_1(z)\, e^{-i\omega t} \quad , \tag{2.16b}$$

where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \alpha^2\right)F = 0 \quad , \tag{2.17}$$

(α is the separation constant) and

$$c_0 \frac{dP_1(z)}{dz} = (N^2-\omega^2)W_1(z) \quad , \tag{2.18a}$$

$$c_0 \frac{dW_1(z)}{dz} = \left(1 - \frac{\alpha^2 c_0^2}{\omega^2 - f^2}\right)P_1(z) \quad , \tag{2.18b}$$

with boundary conditions $-Pc_0=gW$ at $z=0$, and $W=0$ at $z=H$. Then,

$$W_1 = W_0 \sin\left[\beta(z-H)\right] \quad , \tag{2.19a}$$

$$P_1 = P_0 \cos\left[\beta(z-H)\right] \quad , \tag{2.19b}$$

where β, the vertical wave number, is related to frequency ω and to horizontal wave number α by

$$\beta^2 c_0^2(\omega^2-f^2) + (N^2-\omega^2)(\omega^2-f^2-\alpha^2 c_0^2) = 0 \quad . \tag{2.20}$$

On the other hand, the free surface boundary condition gives

$$N^2 - \omega^2 = g\beta \tan\beta H \quad . \tag{2.21}$$

A graph of the simultaneous solution to Eqs. (2.20) and (2.21)[17] is shown in Fig. 1. Case A is $N^2<\alpha^2 c_0^2+f^2$; case B is $N^2>\alpha^2 c_0^2+f^2$. In either case, each solution to Eq. (2.21) intersects the solution to Eq. (2.20) twice, resulting in two sets of normal modes for the system. The dispersion relation for these modes is shown in Fig. 2.[17] The gravitational (internal wave) modes are confined between the rotational and buoyancy frequencies whereas the acoustic modes have frequencies much higher than N. For the sake of clarity, the spacing of the acoustic modes has been greatly reduced and the slopes of the other curves greatly increased in Fig. 2.

What happens in the ocean is this: internal waves move great quantities of water up and down (and also back and forth, but this is irrelevant to sound propagation) at velocities slow enough to essentially maintain the static entropy and salinity gradients. The phase velocities of these internal waves are sufficiently smaller than the sound speed that this motion can be considered incompressible; the

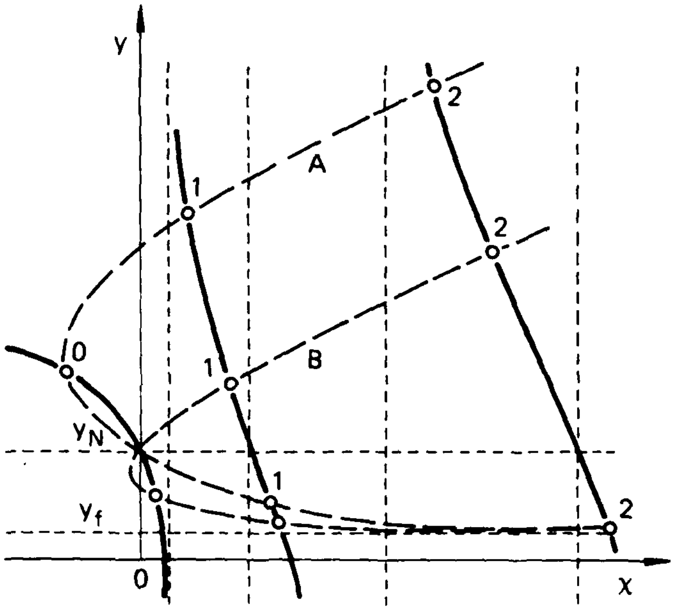


FIGURE 1
DIAGRAM FOR THE SIMULTANEOUS SOLUTION OF THE SIMULTANEOUS EQUATIONS (2.20) AND (2.21).
HERE, $x = (\beta H)^2$, $y = \omega^2 H/g$, $y_N = N^2 H/g$, AND $y_f = f^2 H/g$.
(FIGURE TAKEN FROM REF. 17).

ARL:UT
AS-84-818
CMD - GA
9 - 24 - 84

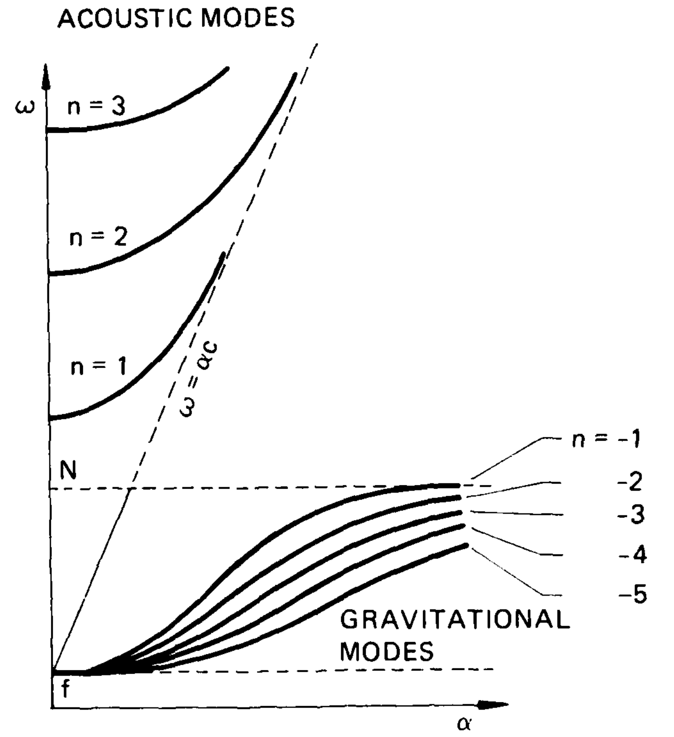


FIGURE 2
DIAGNOSTIC DIAGRAM FOR THE GENERAL CASE OF AN OCEAN OF CONSTANT DEPTH AND CONSTANT N, Γ, c.
(FIGURE TAKEN FROM REF. 17).

ARL:UT
AS-84-819
CMD - GA
9 - 24 - 84

frequencies are so low as to make gravitation and the earth's rotation important. Sound waves are transmitted along at velocities so high that particles under the influence of internal waves seem "frozen". The sound frequencies are so high that gravitation and Coriolis effects affect sound only in the fact that the medium through which it is propagating has been rearranged somewhat. Also, the heat and salt contained in a volume of water displaced by a sound wave don't have time to diffuse into the new environment before the volume element is pushed back again. In the linear regime, sound is assumed to consist strictly of compression waves.

The total particle motion is therefore the sum of rotational, incompressible motion due to the internal waves and irrotational, compressible motion due to sound. Each of these effects is assumed to independently satisfy the linearized hydrodynamic equations; the effect of the internal waves upon the sound is manifested through the equation of state. That is, the sound speed is considered to possess a small perturbation due to the displacement of isodensity surfaces by the internal waves.

We may now feel comfortable in deriving the acoustic wave equation in the usual manner, secure in believing internal wave effects to be contained in the sound speed, with the result

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \text{source terms.} \tag{2.22}$$

Chapter 3

The ocean sound speed is a function of horizontal position $\underline{r}$, depth z, and time t. We take as our model

$$c(\underline{r},z,t) = \bar{c}(z) + \delta c(\underline{r},z,t) \quad , \qquad (3.1)$$

where $\bar{c}(z)$ is the mean sound speed profile and $\delta c(\underline{r},z,t)$ describes the sound speed fluctuations due to internal waves. In general, $\bar{c}(z)$ will vary with horizontal position and time. However, these changes are usually very slow compared with the rate of variation in δc; $\bar{c}(z)$ mainly depends on depth.

A. The Mean (Deterministic) Sound Speed Profile

The mean sound speed profile for the deep ocean is characterized by a sound channel. Due to a negative temperature gradient $\bar{c}(z)$ decreases as depth from the surface increases. This effect is eventually balanced by the weight of the water and $\bar{c}(z)$ reaches a minimum c_0 at the channel axis depth z_0. For $z>z_0$, the sound speed increases with depth due to increasing pressure in the nearly isothermal deep ocean.

Details of this picture vary from ocean to ocean (e.g., Urick[18]). For example c_0 may be found very near the surface in cold Arctic regions, resulting in a sound speed profile which essentially increases throughout the entire water layer. Nearer the equator, in cloudy, windy environments, turbulent mixing may result in an

isothermal layer just below the ocean surface. Within this layer, the sound speed gradient vanishes twice, once near the surface at the base of the "mixed layer" and again deep in the ocean at depth z_0.

We need not consider these details for the study at hand. We do require a model of $\bar{c}(z)$ which reasonably describes the sound speed profile for much of the world's deep ocean environment and has its basis in physical reality. Therefore, the "canonical" sound speed profile derived by Munk[19] is chosen.

Munk's derivation of the sound speed profile is reproduced in Appendix A. Briefly, realistic assumed dependences on temperature and salinity by the sound speed are combined with an exponentially decreasing stratification profile, which is "perhaps the most intrinsic property of the abyssal oceans" (Fig. 3, Munk;[19] also, Refs. 2, 20-21). By "stratification" is meant only that the density changes with depth. The word does not always imply layers but applies to continuous density gradients as well. Although the assumption of exponential stratification is really invalid near the surface (see references cited above; also Eckart[22]) we follow Munk and everybody else in ignoring this. The fluid is statically stable when the buoyancy frequency

$$N(z) \equiv \sqrt{\frac{g}{\rho_0}\left(\frac{\partial\rho_0}{\partial z}\right)_P} = N_0\, e^{-z/B} \qquad (3.2)$$

is real. In Eq, (3.2), B is the scale depth of the stratification, g is the acceleration due to gravity, and $\left(\frac{\partial\rho_0}{\partial z}\right)_P$ is the potential density gradient.

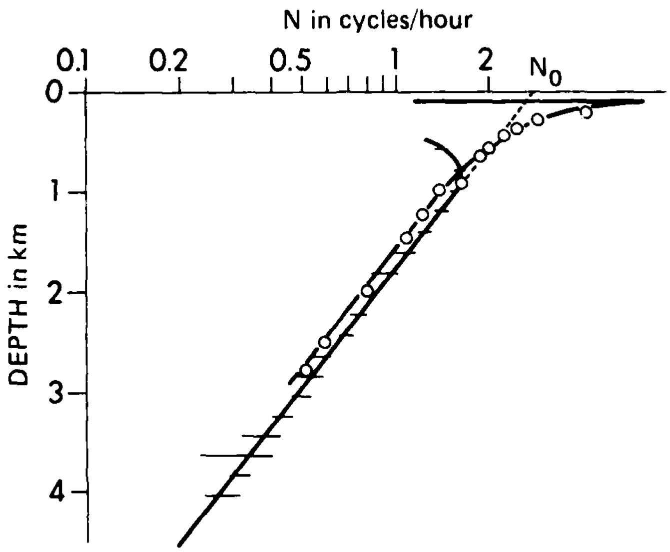


FIGURE 3
THE VÄISÄLÄ (OR BUOYANCY) FREQUENCY $N(z) = [g\rho^{-1}\partial_z\rho_P]^{1/2}$.
HORIZONTAL BARS GIVE RANGE IN VALUES MEASURED OVER THE BISCAY ABYSSAL PLAINS (46°N 8°W) BY PINGREE AND MORRISON.[21] BENEATH THE THERMOCLINES THESE HAVE BEEN FITTED BY $N(z) = N_0 e^{-z/B}$, WITH N_0 = 2.8 cph DESIGNATING THE "SURFACE-EXTRAPOLATED" VALUE OF N(z), AND B = 1.65 km A "BUOYANCY SCALE." IN THE PACIFIC OCEAN (CIRCLES) THE THERMOCLINE IS MUCH SHALLOWER, TYPICALLY 100 m, AND A "SUPER-EXPOTENTIAL" RISE TAKES PLACE BETWEEN 800 m AND THE THERMOCLINES.
(FIGURE TAKEN FROM REF. 19).

ARL:UT
AS-84-820
CMD - GA
9 - 24 - 84

The sound speed channel appears quite naturally at a depth very near the scale depth of the buoyancy frequency. The analytical expression for $\bar{c}(z)$ is

$$\bar{c}(z) = \bar{c}(z_0) \exp\left[\varepsilon\left(\Delta + e^{-\Delta} - 1\right)\right] \quad , \qquad (3.3)$$

where

$$\Delta = \frac{2}{B}(z - z_0) \qquad (3.4)$$

and

$$\varepsilon = \frac{1}{2} B\gamma_A \quad . \qquad (3.5)$$

The quantity γ_A is the fractional sound velocity gradient in an adiabatic isohaline ocean. For real oceans,

$$\bar{c}(z) \sim c_0\left[1 + \varepsilon\left(\Delta + e^{-\Delta} - 1\right)\right] \quad , \qquad (3.6)$$

which is the well-known Munk (canonical) sound speed profile.

B. The Random Sound Speed Perturbation

The sound speed at position $\underline{x}$ (as measured in a three-dimensional coordinate system) and time t is assumed to vary from its mean value $\bar{c}(z)$ by the slow advection of an isodensity surface by internal waves. The advection is slow enough to be quasistatic;

therefore we may write the sound speed fluctuation at position $\underline{x}$ and time t,

$$\delta c(\underline{x},t) \simeq \left(\frac{\partial c}{\partial z}\right)_p \zeta(\underline{x},t) \quad , \tag{3.7}$$

where ζ is the vertical particle displacement due to internal waves. Our problem is now to describe ζ. There are two popular ways of doing this. One method is to make whatever assumptions are necessary to make the hydrodynamic equations describing the internal wave field tractable and use the resulting theoretical description. The second is to take the description of the internal wave field directly from the experimental results without explaining in detail the mechanisms causing the internal wave motion. In the end, we shall choose the second approach, but it is worthwhile to consider the first. The usual approach is to assume that the hydrodynamic equations may be linearized and to neglect shear coupling. The experimental evidence concerning these assumptions will be discussed later. The linearized equations of motion are

$$\rho_0 \frac{\partial \underline{u}_i}{\partial t} + \rho_0(\underline{f}\times\underline{u}_i) = - \nabla p_i + \rho_i g\hat{k} \quad ,$$

$$\frac{\partial \rho_i}{\partial t} + w_i \frac{\partial \rho_0}{\partial z} = 0 \quad , \qquad \nabla\cdot\underline{u}_i = 0 \quad , \tag{3.8}$$

where the subscript i denotes internal wave (as opposed to acoustic) effects and where $w_i = \dot{\zeta}$ is the vertical component of $\underline{u}_i$, the particle velocity due to the internal waves. Given Eqs. (3.8), one can show, for $N^2 >> g^2/c^2$,

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w_i) + N^2(z)\left(\nabla_H^2 w_i\right) + (\underline{f}\cdot\nabla)^2 w_i =$$

$$\frac{-N^2(z)}{g}\left[\frac{\partial^2}{\partial t^2}\frac{\partial w_i}{\partial z} + \underline{f}\cdot\left(\nabla \times \frac{\rho_i g}{\rho_o}\ \hat{k}\right) + \underline{f}\cdot(\underline{f}\cdot\nabla)\underline{u}_i + \frac{\partial}{\partial t}\frac{\partial}{\partial z}(\underline{f}\times\underline{u}_i)\cdot\hat{k}\right], \tag{3.9}$$

where $\nabla_H^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ (see Appendix B). At this point we must make some approximations in order to make this equation tractable. The "traditional approximation", so called by Eckart[23] because all previous treatments of the problem agreed to make it, consists of neglecting products of w_i and f_H, the horizontal component of $\underline{f}$. This approximation is often justifiable. However, as Eckart points out, neglect of this term without also neglecting similar products of f_H and the other velocity components robs the equations of motion of their self-adjointness. The nature of the field equations is thus completely altered and radically different phenomena will be described. Therefore, if $w_i f_H$ is to be neglected, so must the other terms involving f_H. This is clearly invalid near the equator. However, if we adopt the traditional approximation,

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w_i) + N^2(z)\left(\nabla_H^2 w_i\right) + f_z^2 \frac{\partial^2 w_i}{\partial z^2} =$$

$$\frac{-N^2(z)}{g}\left[\frac{\partial^2}{\partial t^2}\frac{\partial w_i}{\partial z} + f_z^2 \frac{\partial w_i}{\partial z}\right] \quad . \tag{3.10}$$

Plane wave solutions to Eq. (3.10) have the form

$$w_i = W(z)\, e^{i(\underline{\alpha}\cdot\underline{r}-\Omega t)} \quad , \tag{3.11}$$

where (dropping the subscript on f)

$$\frac{\partial^2 W(z)}{\partial z^2} + \alpha^2\left[\frac{N^2(z) - \Omega^2}{\Omega^2 - f^2}\right]W(z) = \frac{-N^2(z)}{g}\frac{\partial W(z)}{\partial z} \quad . \tag{3.12}$$

The right side of Eq. (3.12) may be neglected since the ratio of the first to the second derivative of W(z) is no larger than the depth of the ocean[24] and

$$\frac{N^2(z)}{g} H \lesssim 5\times10^{-4} \quad . \tag{3.13}$$

The quantity W(z) is subject to boundary conditions at the surface (z=0) and at the ocean floor (z=H).[16] W(H) is generally

taken to be zero since the ocean floor is considered to be impermeable. Also, since the internal waves in the ocean produce only very small vertical displacements of the free surface,[25] W(0) is generally taken to be zero, too.

Given the boundary conditions and a particular horizontal wave number α, the solutions of Eq. (3.10) form a discrete set of eigenfunctions, each corresponding to an eigenfrequency $\Omega_{j\alpha}$. For real α, these eigenfrequencies will lie in the interval $f < \Omega_{j\alpha} < \max N(z)$. A vertically displaced fluid element will oscillate vertically with the buoyancy frequency N(z). If the displacement is not vertical, the restoring force is less and the fluid element's motion will be elliptical with a reduced frequency. At frequencies just slightly higher than f, the particles will travel in nearly horizontal, circular orbits, the velocity vector rotating anticyclonically (Fig. 4; for more detail see Munk[1] and Phillips[26]).

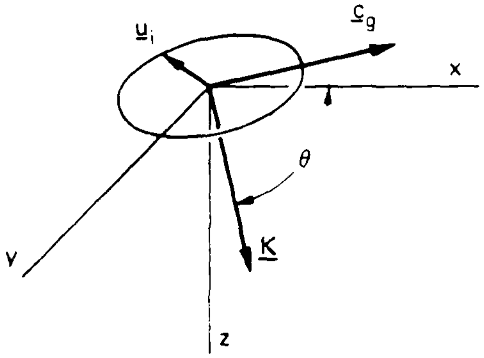
For an exponentially decreasing buoyancy frequency $N(z)=N_0\,e^{-z/B}$, the solutions to Eq. (3.10) are exact,[2] and given by

$$W_j(\alpha,z) = \mathcal{N}\left[J_a(y) - \frac{J_a\big(y(H)\big)}{Y_a\big(y(H)\big)}\,Y_a(y)\right] \quad , \qquad (3.14)$$

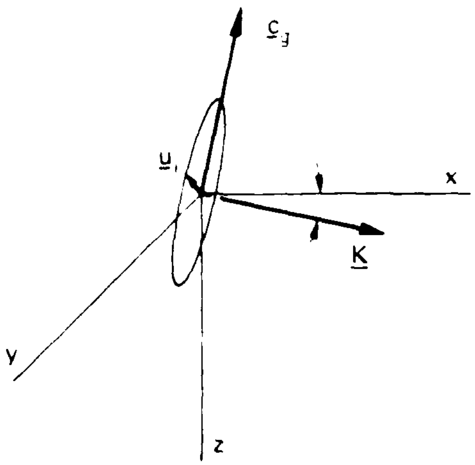
where

$$a = \frac{\alpha B}{\Omega_{j\alpha}}\left(\Omega_{j\alpha}^2 - f^2\right)^{1/2} \quad , \qquad (3.15)$$

$$y(z) = \frac{a}{\Omega_{j\alpha}}\,N(z) \quad . \qquad (3.16)$$



INERTIAL FREQUENCY



BUOYANT FREQUENCY

FIGURE 4
THE WAVENUMBER VECTOR $\underline{K}$ = ($\underline{\alpha}$, β), GROUP VELOCITY $\underline{c}_g$ AND THE HODOGRAPH OF THE PARTICLE VELOCITY $\underline{u}_i$(t) NEAR INERTIAL FREQUENCY (Ω = f+) AND BUOYANT FREQUENCY (Ω = N-), RESPECTIVELY. $\underline{K}$ IS NORMAL TO BOTH $\underline{c}_g$ AND $\underline{u}_i$
(FIGURE TAKEN FROM REF. 1)

ARL:UT
AS-84-821
CMD - GA
9 - 24 - 84

$\mathcal{N}$ is a normalization constant and J_a, Y_a are Bessel functions of the first and second kind. The eigenfrequencies $\Omega_{j\alpha}$ are determined by the upper boundary condition $W_j(\alpha,z)=0$ at $z=0$.

Now we may represent the vertical displacement of an isodensity surface by a superposition of internal wave modes,[27]

$$\zeta(\underline{x},t) = \sum_j \int d\underline{\alpha} G_j(\underline{\alpha}) W_j(\alpha,z)\, e^{i(\underline{\alpha}\cdot\underline{r}-\Omega_{j\alpha}t)} \quad , \tag{3.17}$$

where the quantities $G_j(\underline{\alpha})$ are complex Gaussian random variables satisfying

$$\langle G_j(\underline{\alpha}) \rangle = 0$$

$$\langle G_j(\underline{\alpha}) G^*_{j'}(\underline{\alpha}') \rangle = P_j(\underline{\alpha}) \delta(\underline{\alpha}-\underline{\alpha}') \delta_{jj'} \; . \tag{3.18}$$

Before discussing the form of the spectrum $P_j(\underline{\alpha})$, it is necessary to say a few words about the above analysis. The derivation of Eq. (3.9) neglects several non-negligible effects: forcing, shear dissipation, nonlinear coupling.[28-29] It is doubtful that well-defined internal wave modes, other than perhaps the gravest mode, will have time to establish themselves since vertical propagation times of the wave field are comparable to typical interaction times. This suggests that it might be better to replace the discrete sum in Eq. (3.17) by an integral over a continuous frequency range.

Observations[30-32] tend to support this latter notion; propagating waves dominate at low frequencies, standing modes may dominate at the highest frequencies ($j\sim 1$), and both descriptions are adequate in between. We therefore choose the second method to describe the internal wave field, specifically the statistical description of ζ, relying heavily on experimental work which will be referenced as we go along. In this latter description, one assumes a random state with a continuum of energy in frequency/wave number space. Here, a difficulty arises in describing the spectrum since the medium is not depth homogeneous. Nevertheless, the quantity

$$\beta = \alpha \left(\frac{N^2(z) - \Omega^2}{\Omega^2 - f^2} \right)^{1/2} \tag{3.19}$$

can be considered a local vertical wave number, in the WKBJ sense. Thus, for $\Omega < N(z)$, we can use Eq. (3.15) as the frequency/wave number dispersion relation.

The continuum description used here was introduced by Garrett and Munk,[2,33] modified by experiment,[32,34,35] and streamlined by the analytical work of Desaubies.[36]

Any function of two (three-dimensional) positions $\underline{x}=(\underline{r},z)$ and $\underline{x}'=(\underline{r}',z)$ relative to some fixed coordinate system can be written in terms of their average position $(\underline{x}'+\underline{x})/2$ and their difference in position $\Delta\underline{x}=(\underline{x}-\underline{x}')$. The covariance $\langle \zeta(\underline{r},z)\zeta(\underline{r}',z') \rangle$ will depend on the average position only through the average vertical position

$\eta=(z+z')/2$ (Ref. 37). Thus, if $(\underline{\alpha},\beta)$ is the wave number vector conjugate to $(\underline{r}-\underline{r}', \mathcal{z})$, where $\mathcal{z}=|z-z'|$, we may write

$$<\zeta(\underline{r},z)\zeta(\underline{r}',z')> = \int_0^\infty d\beta \int d\underline{\alpha} F_\zeta(\underline{\alpha},\beta;\eta)\cos\beta\mathcal{z}\; e^{i\underline{\alpha}\cdot(\underline{r}-\underline{r}')}, \qquad (3.20)$$

where $F_\zeta(\alpha,\beta;\eta)$ is normalized such that

$$\int d\underline{\alpha}d\beta F_\zeta(\underline{\alpha},\beta;\eta) = <\zeta^2(\underline{x}-\underline{x}',\eta)> = <\zeta^2(\eta)> \quad . \qquad (3.21)$$

If $F_\zeta(\alpha,\beta;\eta)$ depends on the horizontal wave number $\underline{\alpha}$ only through its magnitude (that is, if the medium is statistically isotropic in the horizontal plane), we introduce

$$P_\zeta(\alpha,\beta;\eta) = 2\pi\alpha F_\zeta(\underline{\alpha},\beta;\eta) \qquad (3.22)$$

such that

$$\int_0^\infty d\beta\int_0^\infty d\alpha\, P_\zeta(\alpha,\beta;\eta) = \int_0^{2\pi} d\theta\int_0^\infty d\beta\int_0^\infty d\alpha\alpha F_\zeta(\underline{\alpha},\beta;\eta) = <\zeta^2(\eta)> . \qquad (3.23)$$

If $\underline{x}=(r,\varphi,z)$ and $\underline{x}'=(r',\varphi',z')$ are two positions in some fixed (cylindrical) coordinate system, we may write Eq. (3.20) in what will prove to be a more convenient form:

$$<\zeta(\underline{r},z)\zeta(\underline{r}',z')> = \frac{1}{2\pi}\int_0^\infty d\beta\int_0^\infty d\alpha\int_0^{2\pi} d\theta P_\zeta(\alpha,\beta;\eta)\cos\beta\mathcal{z}$$

$$\times e^{i\alpha\left[r\cos(\varphi-\theta)-r'\cos(\varphi'-\theta)\right]} \quad . \qquad (3.24)$$

Finally, we may transform from (α,β) space to (α,Ω) space by use of the dispersion relation (3.19):

$$P_\zeta(\alpha,\Omega;\eta) = P_\zeta(\alpha,\beta(\alpha,\Omega);\eta)\,\frac{\partial\beta}{\partial\Omega} = NA(\alpha)O(\Omega) \quad , \tag{3.25}$$

where, according to the references cited above Eq. (3.20),

$$A(\alpha) = \frac{\alpha_\star}{\alpha^2 + \alpha_\star^2} \quad , \tag{3.25a}$$

$$O(\Omega) = \frac{4f}{\pi}\,\frac{(\Omega^2-f^2)^{1/2}}{\Omega^3} \quad , \tag{3.25b}$$

and

$$N = \frac{2}{\pi} <\zeta^2(\eta)> \tag{3.25c}$$

ensure that

$$\int_f^{N(\eta)} d\Omega \int_0^\infty d\alpha P(\alpha,\Omega;\eta) = <\zeta^2(\eta)> \tag{3.26}$$

to order $f/N(\eta)$. In most cases of interest, min $N(\eta) >> f$. The quantity $\alpha_\star = t(\Omega^2-f^2)^{1/2}$ is a number characteristic of the bandwidth; t is an experimentally determined constant on the order of 10^{-3} - 10^{-4} h/m.

The model presented in Eqs. (3.25) depends essentially on a WKBJ approximate solution to Eq. (3.12) and is therefore not very good

for frequencies near $N(\eta)$ (Fig. 5; taken from Ref. 35). In particular, the spectrum shows a pronounced hump just below the buoyancy frequency, followed by a sharp cutoff.

This hump is also found in the internal wave experiment (IWEX) data[32,34] and in previous measurements by Voorhis[38] and Gould.[39] Vertically propagating waves with frequency Ω are reflected at the turning depth (where $\Omega = N(z)$); each of these wavefunctions has therefore an inflection point at that depth. Waves which destructively interfere in shallower water are therefore locked in phase near the turning depth.[40] A Langer method of treating the internal wave field[40,41] has been shown to describe the observed frequency spectrum very well (Fig. 6). However, most of the energy in the internal waves is contained in frequencies near f (Eq. 3.24b has a sharp peak at $\Omega = f\sqrt{1.5}$). Thus the simpler WKBJ description is adequate for our purposes.

The description is not yet complete; we still have to specify $\langle \zeta^2(\eta) \rangle$. Garrett and Munk[2] took the WKBJ solution to Eq. (3.14) and "depth averaged over many wiggles" to find that, to some approximation, $\langle \zeta^2(\eta) \rangle \propto N^{-1}(\eta)$. Whether or not this was a legitimate procedure, the observations support this result. When the autospectra were measured at different depths (data taken in IWEX) and the "WKBJ normalized", i.e., multiplied by $N(z)/N(z_r)$ where z_r is a reference depth, the variation in the spectral levels were reduced to

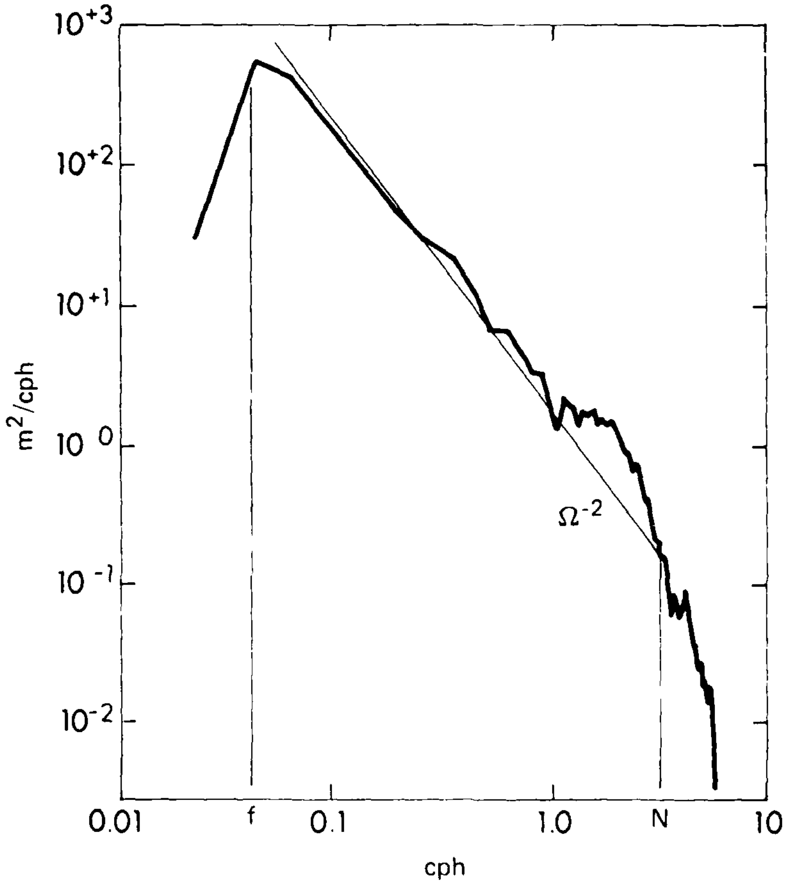


FIGURE 5
POWER SPECTRUM OF VERTICAL DISPLACEMENT OF AN ISOTHERM
(FIGURE TAKEN FROM REF. 35)

ARL:UT
AS-84-822
CMD - GA
9 - 24 - 84

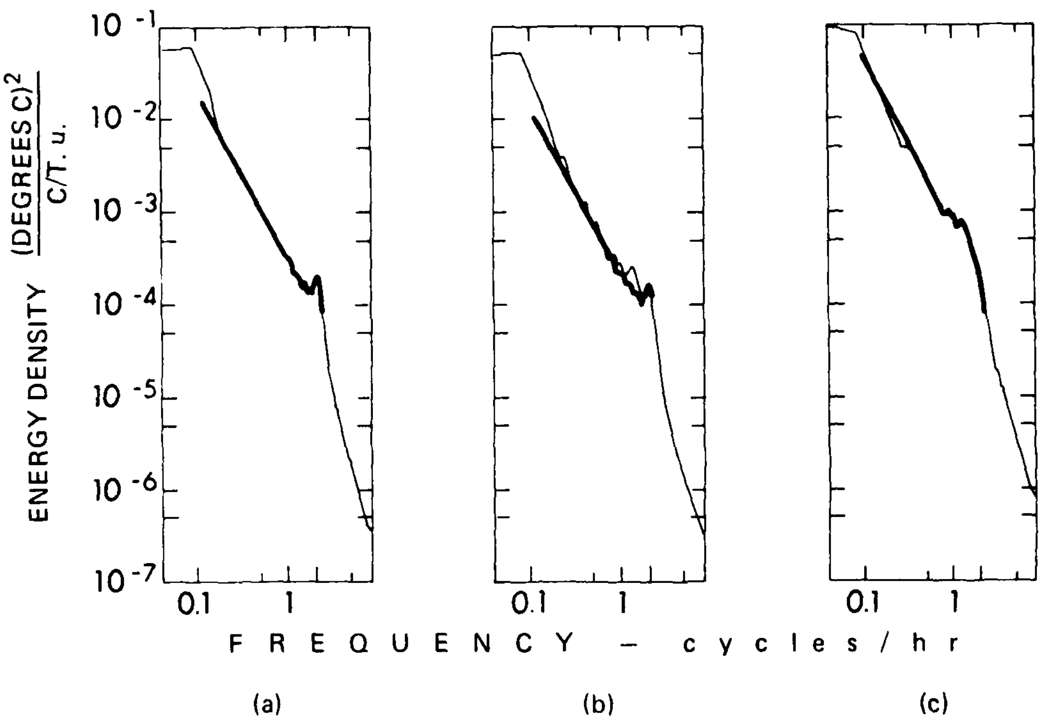


FIGURE 6
IT IS ASSUMED THAT TEMPERATURE FLUCTUATIONS ARE RELATED TO DISPLACEMENT FLUCTUATIONS BY $\delta T = \left(\frac{\partial T}{\partial z}\right)_0 \zeta$.
FOR DETAILS OF THE DATA SEE REF. 40.
LIGHT LINE: DATA. HEAVY LINE: LANGER APPROXIMATION.
(FIGURE TAKEN FROM REF. 40).

ARL:UT
AS-84-823
CMD - GA
9 - 24 - 84

some 10%[32,34] (Fig. 7) except near the buoyancy frequency, where the WKBJ approximation breaks down. Again, we ignore the region of invalidity.

Our model of the sound speed fluctuations is now finished. Relating these sound speed fluctuations to the displacement correlations gives

$$\frac{\langle \delta c(r,\varphi,z)\,\delta c(r',\varphi',z')\rangle}{\bar{c}(z)\bar{c}(z')} = \frac{4}{\pi^3}\left(\frac{\mu}{g}\right)^2 \langle \zeta^2(o)\rangle \; N_o f \; \frac{N^2(z)\; N^2(z')}{N(\eta)}$$

$$\int_f^{N(\eta)} d\Omega \int_{-\infty}^{\infty} d\alpha \; \frac{\alpha_*}{(\alpha^2+\alpha_*^2)} \; \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3} \cos(\alpha \mathcal{Z} X(\Omega))$$

$$\int_o^{\pi/2} d\theta \left[e^{i\alpha[r\cos(\varphi-\theta)-r'\cos(\varphi'-\theta)]} + e^{i\alpha[r\cos(\varphi+\theta)-r'\cos(\varphi'+\theta)]} \right] . \tag{3.27}$$

where we have used the fact that the integrand is even in α and where

$$X(\Omega) = \left(\frac{N^2 - \Omega^2}{\Omega^2 - f^2}\right)^{1/2} , \tag{3.28a}$$

$$\langle \zeta^2(\eta)\rangle = \langle \zeta^2(o)\rangle \; \frac{N_o}{N(\eta)} . \tag{3.28b}$$

For completeness, we also give the spectrum in (β,Ω) space:

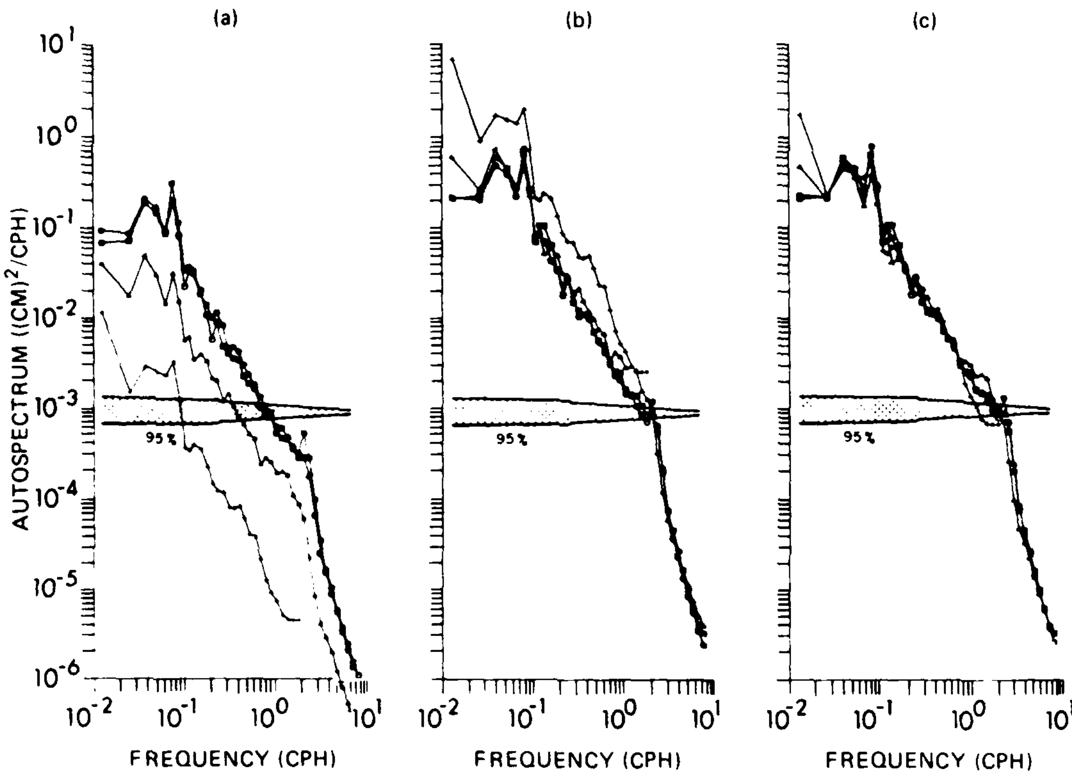


FIGURE 7

TEMPERATURE AND UP AUTOSPECTRA; *(a)* TEMPERATURE; *(b)* UP (*i.e.*, VERTICAL DISPLACEMENT BASED ON THE MEASURED MEAN VERTICAL TEMPERATURE GRADIENT); (c) WKBJ-NORMALIZED UP AUTOSPECTRA. DIFFERENT SYMBOLS DENOTE DIFFERENT DEPTHS:

SQUARE: 604 m   CIRCLE: 731 m   TRIANGLE: 1023 m   CROSS: 2050 m

(FIGURE TAKEN FROM REF 34)

ARL:UT
AS-84-824
CMD - GA
9 - 24 - 84

$$P_\zeta(\beta,\Omega) = P_\zeta(\alpha(\beta,\Omega),\Omega)\,\frac{\partial\alpha}{\partial\beta} = N\mathcal{B}(\beta)\mathcal{O}(\Omega) \quad , \qquad (3.29)$$

where

$$\mathcal{B}(\beta) = \frac{\beta_*}{\beta^2 + \beta_*^2} \quad , \qquad (3.30a)$$

and

$$\beta_* = t(N^2(\eta)-\Omega^2)^{1/2} \quad . \qquad (3.30b)$$

Chapter 4

In this chapter we review normal mode theory for a horizontally stratified, deterministic ocean. In particular, we consider a single water layer of depth H overlying a fluid half-space of constant compressional velocity c_p, which will be larger than the sound speed minimum c_0. In the following analysis we consider a time-harmonic source. It is therefore more convenient to describe sound propagation by means of a velocity potential $\psi(\underline{r},z)$ such that the particle velocity $\underline{u}$ due to the acoustic wave is

$$\underline{u} = \nabla\psi(\underline{r},z)\, e^{-i\omega t} \quad . \tag{4.1}$$

In this formalism, the acoustic pressure p is simply related to ψ by

$$p = i\omega\rho\psi\, e^{-i\omega t} \quad , \tag{4.2}$$

where ρ is the mass density of the medium. Changes in density with depth are closely related to changes in the sound speed, as we have seen. Over the depth of the ocean, the sound speed and density change very little compared with changes in ψ. Thus, we may write (from Eq. (2.22)) a wave equation for ψ.

$$\nabla^2\psi + \frac{\omega^2}{c^2}\psi = Q(\underline{r},z) \quad , \tag{4.3}$$

where $Q(\underline{r},z)\frac{e^{-i\omega t}}{i\omega\rho}$ represents the source terms which are the right side of Eq. (2.22). The simple source we shall consider is the time-harmonic point source

$$Q(\underline{r},z) = -4\pi Q_0 \delta(\underline{x}-\underline{x}_0) \quad , \tag{4.4}$$

where $\underline{x}_0$ is the position of the source. We shall choose our coordinate system such that $\underline{x}_0 \equiv (\underline{0},z_0)$; the water-air interface is at z=0.

In the deterministic medium chosen here, c varies only with depth so that

$$\psi(\underline{r},z) = \phi(z)F(r) \quad . \tag{4.5}$$

That is, there is no azimuthal dependence and ψ separates into a product of functions which satisfy ordinary differential equations. In particular

$$\frac{d^2\phi}{dz^2} + \left(\frac{\omega^2}{c^2} - k^2\right)\phi = 0 \quad , \tag{4.6}$$

where k is the separation constant.

The boundary condition that the normal component of stress be continuous across an interface implies that $\rho\phi$ is continuous across such an interface and, since u_z must also be continuous across an interface, so is $\frac{d\phi}{dz}$. Since the mass density of air is much smaller

than that of water, we may set $\phi(0)=0$. Finally, only downgoing waves are permitted as $z\to\infty$.

If k is greater than $\frac{\omega}{c_p}$, we have a family of mode functions which exponentially decay as $z\to\infty$. These modes correspond to waveguide phenomena in which energy is trapped by total reflection.[42] For $k > \frac{\omega}{c_p}$, the eigenvalue spectrum is discretely indexed[43] and, if there are M discrete eigenvalues,

$$\frac{\omega}{c_o} > k_1 > k_2 > k_i > \ldots k_M > \frac{\omega}{c_p} \quad . \tag{4.7}$$

The eigenfunction $\phi_i(z)$ corresponding to the ith discrete eigenvalue will have i zeroes in the depth interval $[0,H)$.

For $k<\frac{\omega}{c_p}$, the spectrum of k is continuous. The modes corresponding to this continuous spectrum are generally neglected because they are attenuated much more rapidly than the discrete modes, due to both the geometric spreading and the volume absorption which is generally smaller in the water layer than in any underlying layers.

It is a simple matter to show that the discrete eigenfunctions ϕ_n are orthogonal.[44] We have also normalized them so that

$$\int_0^\infty dz\rho\phi_n\phi_m = \delta_{nm} \quad , \tag{4.8a}$$

$$\int_0^\infty dz\rho\phi(k)\phi(k') = \delta(k-k') \quad , \tag{4.8b}$$

$$\int_0^\infty dz\rho\phi_n\phi(k) = 0 \quad . \tag{4.8c}$$

Turning to the range functions, we find upon imposing a Sommerfeld condition that

$$F(r) \propto H_0^{(1)}(kr) \quad , \tag{4.9}$$

where $H_0^{(1)}(kr)$ is a zeroth-order Hankel function of the first kind. Thus

$$\psi(\underline{r},z) = i\pi Q_0\rho(z_0)\left\{\sum_{n=1}^{M}\phi_n(z)\phi_n(z_0)H_0^{(1)}(k_nr) + \int_0^{\omega/c_p} dk\phi(z,k)\phi(z_0,k)H_0^{(1)}(kr)\right\} \quad . \tag{4.10}$$

(For more detail see Ref. 45). From now on we follow established practice in dropping the continuous contribution.

The normalization (Eq. 4.8) has a very specific physical meaning. Suppose we have an arrangement of sources such that only one of the discrete normal modes, say the nth mode, and none of the others is excited. Then, the acoustic Poynting vector is given by[46]

$$\underline{j} = \frac{1}{4}\left(p\underline{u}^* + \underline{u}p^*\right) = \frac{i\omega\rho}{4}\left(\psi(\nabla\psi)^* - (\nabla\psi)\psi^*\right) =$$

$$\frac{i\pi^2\omega}{4} Q_o^2\rho^2(z_o)\rho(z)\phi_n^2(z_o)\left[\phi_n(z)H_o^{(1)}(k_n r)\left(\nabla[\phi_n(z)H_o^{(1)}(k_n r)]\right)^*\right.$$

$$\left. - \nabla\left(\phi_n(z)H_o^{(1)}(k_n r)\right)\phi_n(z)H_o^{(2)}(k_n r)\right] \quad . \tag{4.11}$$

The integral of $\underline{j}$ over depth will give the amount of intensity passing through a line at horizontal position $\underline{r}$.

$$\int_o^\infty dz\underline{j} = \hat{r}\,\frac{\pi}{r}\,Q_o^2\rho^2(z_o)\phi_n^2(z_o)\omega\int_o^\infty dz\rho(z)\phi_n^2(z) \quad . \tag{4.12}$$

For this cylindrically symmetric case, the intensity passing through a cylinder of radius r is thus

$$I_r = 2\pi^2 Q_o^2\rho^2(z_o)\phi_n^2(z_o)\omega\int_o^\infty dz\rho(z)\phi_n^2(z) \quad . \tag{4.13}$$

If we define range functions

$$B_n(r) = i\pi Q_o\rho(z_o)\phi_n(z_o)H_o^{(1)}(k_n r) \quad , \tag{4.14}$$

then the velocity potential due to a point source is

$$\psi = \sum_n B_n(r)\phi_n(z) \quad , \tag{4.15}$$

and the acoustic Poynting vector is

$$\underline{j} = \hat{r}\,\frac{\pi}{r}\,Q_o^2\rho_o^2(z_o)\omega \sum_n \phi_n^2(z_o)\rho(z)\phi_n^2(z)$$

$$+ \frac{i\omega\rho}{4} \sum_{n,m}{}' \left\{ \hat{r}\phi_n(z)\phi_m(z)\left[B_n(r)\left(\frac{dB_m(r)}{dr}\right)^* - B_n^*(r)\left(\frac{dB_m(r)}{dr}\right)\right]\right.$$

$$\left. + \hat{z}\phi_n(z)\,\frac{d\phi_m(z)}{dz}\left[B_n(r)B_m^*(r) - B_n^*(r)B_m(r)\right]\right\} . \qquad (4.16)$$

The first term on the right hand side of Eq. (4.16) is the sum of the intensities of the individual modes had each been excited by itself; this term may therefore be called a "self" energy term. The restricted sum represents the intermodal interference and is truly an interaction term. Note that the self-energy term travels in a purely radial direction; the intermodal interference is thus solely and explicitly responsible for the bending of ray paths in the ocean.

Chapter 5

Although the spectral density of the sound speed fluctuations depends only on the magnitude of the horizontal wave number, and not on its direction (Eq. 3.25), there will certainly be random azimuthal fluctuations in the sound speed. These fluctuations will in turn manifest themselves through azimuthal variations in the acoustic pressure.

Previous studies in acoustic normal mode theory[6-8] have neglected this azimuthal variation in the acoustic field. Kohler and Papanicolaou[8] did not specifically consider internal waves as the perturbation mechanism; they simply assumed a priori that the sound speed fluctuations did not depend on azimuth. Dozier and Tappert,[6] who used the theoretically derived description of $\zeta(\underline{x},t)$(Eq. 3.17), integrated the spectral density of the sound speed fluctuations along the direction perpendicular to the line connecting the acoustic point source and the receiver. Therefore, the function they employed as the spectrum is actually a spectral density of wave number components in the direction of propagation. By then approximating the function $W_j(\alpha,z)$ in Eq. (3.17) by $W_j(\alpha_1,z)$, where α_1 is the component of $\underline{\alpha}$ in the direction of propagation, all vector dependence by ζ on horizontal position is replaced by a scalar dependence on range. Let α_2 be the component of $\underline{\alpha}$ perpendicular to $\underline{\alpha}_1$. The justification for the replacement of $W_j(\alpha,z)$ by $W_j(\alpha_1,z)$ is that "this is a good approximation for small α_2 and for large α_2 (and hence large α) the

spectrum will be low anyway." The key words here are "small" and "large" since they immediately raise the question of comparison. Clearly the comparison is between α_1 and α_2; this approximation is good for α_2 "small" compared with α_1, so we would expect the approximation to be invalid near the peak of the spectrum for α_1 and α_2 both small and about the same size. Thus large errors in the description of the acoustic field may or may not result.

The most disturbing effect of this assumption is to preclude scattering of energy among the azimuthal acoustic modes; the energy is constrained to remain in the azimuthally symmetric mode. It is the purpose of this chapter to discuss this case but without making any assumptions about the relative sizes of α_1 and α_2. That is, we allow azimuthal fluctuations in the sound speed but assume that azimuthal coupling in the acoustic field may be neglected. We will thus employ the continuum description of the internal wave field discussed in Chapter 3. In addition, we do not make the "random phase approximation"; different depth modes may indeed be correlated. We do not expect drastic departures from the results obtained by Dozier and Tappert[6] for this case where we have artificially prohibited azimuthal coupling. Still, this simpler case is useful in examining many of the physical phenomena, which may be somewhat obscured by the added complexity of the azimuthal coupling, and in introducing the method of analysis which will also be employed later when we do not ignore this azimuthal coupling.

A. The Coupled Mode Equations

Consider a flat slab of seawater of thickness H overlying a substrate. The velocity potential $\Psi(r,z,t)$ due to a time-harmonic point source of angular frequency satisfies the scalar wave equation (cylindrical coordinates)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{\partial^2\Psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}=-4\pi Q_o\delta(\underline{x}-\underline{x}_o)\ e^{-i\omega t} \quad . \tag{5.1}$$

Because the sound speed fluctuations vary with time and range, variables do not exactly separate. However, sound can travel a distance equal to the correlation range of the fluctuations (about 10 km) in travel times on the order of 5-10 sec, much less than the shortest period of the internal waves $N_o^{-1} \sim 20$ minutes. The internal wave field is thus considered "frozen" with respect to the acoustic field. We still have the range variations but since $\left|\frac{\delta c}{\bar{c}}\right| << 1$ (see Eq. 3.1) we can expand Ψ in terms of the normal modes $\phi_n(z)$ of the deterministic sound speed profile $\bar{c}(z)$:

$$\frac{d^2\phi_n(z)}{dz^2}+\frac{\omega^2}{\bar{c}^2(z)}\phi_n(z)=k_n^2\phi_n(z) \tag{5.2}$$

with the normalization (4.8a) and where the relative densities of air and water allow us to take

$$\phi_n(0)=0 \quad . \tag{5.3}$$

If the substrate is rigid,

$$\frac{d\phi_n(H)}{dz} = 0 \quad . \tag{5.4}$$

If, on the other hand, the substrate is an isovelocity fluid with sound speed c_p,

$$\phi_n(H) = \mathcal{N}_n \frac{\rho_s}{\rho_w} \exp\left[\left(k_n^2 - \frac{\omega^2}{c_p{}^2}\right)^{1/2} (H-z)\right] \quad , \tag{5.5}$$

where $\mathcal{N}_n$ is determined by the normalization, ρ_w is the water density at z=H and ρ_s is the substrate density at z=H. We do not consider any other boundary conditions in this study. Thus

$$\Psi(r,z,t) = \psi(r,z)\, e^{-i\omega t} = \sum_{n=1}^{M} B_n(r)\phi_n(z) \frac{e^{-i\omega t}}{\sqrt{r}} \quad , \tag{5.6}$$

where the cylindrical spreading is made explicit.

By expanding our velocity potential in terms of discrete normal modes, we neglect any continuous spectrum of modes. In the rigid bottom and pressure release bottom cases, there are an infinite number of discrete modes. Where real bottom cases are concerned, M is finite and there also exists the continuous spectrum of modes without which the normal mode set is not complete. The traditional assumption that this continuous spectrum does not contribute greatly to the

acoustic field is made here. Therefore, given that we know $B_n(r)$ and $\phi_n(z)$, Eq. (5.6) may not be strictly true but it is certainly a good approximation. A more important error incurred by this approximation occurs in the calculation of $B_n(r)$ since mode coupling into the continuous spectrum, not considered here, may be an important loss mechanism. Kohler and Papanicolaou[8] have advanced a formalism to account for this coupling and an application of this formalism to sound speed fluctuations caused by internal waves would be interesting future work.

All of the randomness in the acoustic field can now be found in the range functions. Since $|\delta c| << \bar{c}$,

$$\frac{\omega^2}{c^2} \simeq \frac{\omega^2}{\bar{c}^2}\left(1 - \frac{2\delta c}{\bar{c}}\right) \quad . \tag{5.7}$$

Employing the orthonormality of the depth functions, we find

$$\frac{d^2 B_n(r)}{dr^2} + \left(k_n^2 + \frac{1}{4r^2}\right) B_n(r)$$

$$= 2\omega^2 \sum_m \int_0^H dz\rho(z)\phi_n(z) \frac{\delta c(r,z)}{\bar{c}^3(z)} \phi_m(z) B_m(r) \quad . \tag{5.8}$$

Now we need to choose initial conditions for the $B_n(r)$. If there were no perturbations due to internal waves, the functions $B_n(r)$ would be the expression given by Eq. (4.14) multiplied by $\sqrt{r}$ (see Eq. 5.6).

The farfield form of the Hankel function is accurate within 4% for $k_n r > 5$ and the accuracy increases rapidly with argument. The deviations from this functional form due to the internal waves is definitely a long range effect; the correlation range of the internal waves is about 10 km.[2,6,47] It is therefore convenient to separate the ocean into two regions: a "nearfield" and a "farfield". The nearfield will consist of the region interior to a cylindrical boundary centered at the source with radius $r_o = \frac{10}{k_M}$ (see Eq. 4.7). Thus, even if $k_M \sim 10^{-2}$ m^{-1}, an order of magnitude smaller than any of the eigenvalues we consider in this study, r_o is still only about 10% of the correlation range of the fluctuations. Within this region, we assume that $B_n(r)$ is given by Eq. (4.14) multiplied by $\sqrt{r}$.

In the farfield ($r > r_o$), we neglect the $1/4r^2$ term since this is less than 1% of k_n^2. $B_n(r)$ will satisfy Eq. (5.8) with initial condition

$$B_n(r_o) = i\sqrt{\frac{2\pi}{k_n}}\, Q_o \rho(z_o) \phi_n(z_o)\, e^{i(k_n r_o - \pi/4)} \quad . \tag{5.9}$$

In terms of forward and backward traveling waves

$$B_n(r) = \frac{1}{\sqrt{k_n}} \left(A_n^+ \, e^{ik_n r} + A_n^- \, e^{-ik_n r} \right) \quad . \tag{5.10}$$

The quantities A_n^+ and A_n^- are complex random functions and we are free to impose one relation between them. We choose

$$e^{ik_n r} \frac{dA_n^+}{dr} - e^{-ik_n r} \frac{dA_n^-}{dr} = 0 \quad . \tag{5.11}$$

so that Eq. (5.8) becomes

$$\frac{dA_n^+}{dr} = -i\omega^2 \sum_m \frac{1}{\sqrt{k_n k_m}} \int_0^H dz \rho(z) \phi_n(z) \frac{\delta c}{c^3} \phi_m(z) \left(A_n^+ e^{i(k_m - k_n)r} + A_n^- e^{-i(k_n + k_m)r} \right) \quad . \tag{5.12}$$

We now make the traditional "forward scattering approximation" by assuming that the rapidly oscillating terms on the right are statistically unimportant. Also called the "parabolic approximation" because of the form of Eq. (5.12), this assumption is equivalent to considering the small-angle scattering much more likely than large vertical deflections. The evidence for this is mainly experimental.[48,49] At any rate, the size of $(k_n - k_m)$ is generally several orders of magnitude smaller than $(k_n + k_m)$. It will be shown, consistent with the results of Dozier and Tappert[6] and of Kohler and Papanicolaou,[8] that the effect of the argument in the exponential is to pick out from the internal wave field scatterers

whose horizontal vector wave numbers have a component the size of the arguments of the exponentials in Eq. (5.12). Since the internal wave spectrum is heavily weighted toward small horizontal wave numbers, it is much easier to find scatterers with horizontal wave numbers on the order of (k_n-k_m) than those with horizontal wave number (k_n+k_m). We therefore ignore the backscattered wave and (dropping the plus sign, consistent with A_n^- being initially zero),

$$\frac{dA_n}{dr} = i\varepsilon \sum_m V_{nm} A_m \quad , \tag{5.13}$$

where

$$V_{nm} = \frac{-\omega^2}{\varepsilon\sqrt{k_n k_m}} \int_0^H dz\rho(z)\phi_n(z)\, \frac{\delta c}{\bar{c}^3}\, \phi_m(z)\, e^{i(k_m-k_n)r} \quad . \tag{5.14}$$

The quantity A_n is the envelope of the rapidly oscillating $\frac{e^{ik_n r}}{\sqrt{r}}$ term and would be constant in a deterministic medium. Since we estimate that changes of order unity will occur in A_n over a range of about a correlation range of the fluctuations ($C_R \sim 10$ km) we have introduced the smallness parameter ε in order to make V_{nm} of the same order as $1/C_R$. Our task now lies in the estimation of ε.

From Eq. (5.14), if $V_{nm} \sim 1/C_R$, $k_n \sim k_m \sim \omega/c_o$, and $\bar{c} \sim c_o$, then

$$\frac{1}{C_R} \sim \left| \frac{-\omega}{\varepsilon c_o} \int_0^H dz\rho(z)\phi_m(z)\, \frac{\delta c}{\bar{c}}\, \phi_n(z) \right| \quad . \tag{5.15}$$

If we now replace $\left(\frac{\delta c}{\bar{c}}\right)$ by the depth average of $<\left(\frac{\delta c}{\bar{c}}\right)^2>^{1/2}$, and also assume that the mode functions contribute very little for z>H,

$$\varepsilon \sim \frac{\omega C_R}{c_0}\left(\frac{B}{3H}\right) < \left(\frac{\delta c}{\bar{c}}\right)^2_{z=0} >^{1/2} \quad . \tag{5.16}$$

Dozier and Tappert[6] estimate $<\left(\frac{\delta c}{\bar{c}}\right)^2_{z=0}>^{1/2} \sim 5.6\times10^{-4}$, so

$$\varepsilon \sim (4.04 \times 10^{-4})\omega \quad . \tag{5.17}$$

The pertubation theory is thus seen to be a low frequency theory and should break down for $\frac{\omega}{2\pi}$ greater than a couple of hundred Hertz.

The expression (5.16) for ε reveals that our smallness parameter is really a product of two competing considerations, the size of the sound speed fluctuations and also their coherence. The quantity $\left(\frac{B}{3H}\right)<\left(\frac{\delta c}{\bar{c}}\right)^2_{z=0}>^{1/2}$ is an estimate of how big the fluctuations are compared with the deterministic sound speed. Clearly, if a sizable fraction of the sound speed is due to the fluctuations, then Eq. (5.7) will not be valid. On the other hand, the quantity $\frac{\omega C_R}{c_0}$ compares the correlation range of the sound speed fluctuations with the acoustic wavelength. If, in the horizontal plane, the characteristic length C_R of the sound speed fluctuations is only a couple of acoustic wavelengths long, the sound speed fluctuations are not strongly correlated, and the acoustic wave will not travel very far without being effectively sped up and slowed down several times by the random effects on the phase. As the phases get more and more

mixed up, the acoustic wave will, in a sense, forget where it has been. On the other hand, a large C_R means that internal wave effects on acoustic propagation will be more systematic. These effects will then tend to accumulate and knowledge of past history will be more important than in a system where the sound speed fluctuations are the same size but less correlated. Since the correlation range is many times the acoustic wavelength, this pertubation theory is valid only because the sound speed fluctuations are so small.

B. The Statistical Behavior of the Range Functions

1. Interesting Moments

The randomness in the sound speed translates into randomness in the velocity potential and any function $f(\psi(r,z))$ of the velocity potential. We are interested in moments of these random functions.

The quantity $\langle\psi(r,z)\rangle$ is definitely not the value of ψ in the absence of any perturbation. If Nature had simply added a random term to the wave equation rather than multiplied it by the state of the system, then first moments of quantities like the acoustic pressure would indeed be their deterministic values, although higher moments would be somewhat more complicated. However, the nature of our problem is explicitly multiplicative and it is thus of some interest to calculate the first moment of $\psi(r,z)$.

In the perturbative method used here, we will calculate moments up to lowest non-vanishing order in ε. That is, any average quantity can be ordered with ε:

$$<f\Big(\mu(r,z)\Big)> = f_0 + \varepsilon <f_1> + \varepsilon^2 <f_2> + \varepsilon^3 <f_3> + \ldots \quad , \tag{5.18}$$

where f_0 is the initial value of f. If, for example, $\varepsilon <f_1>$ vanishes, we will calculate expressions for $<f>$ accurate to $O(\varepsilon^3)$.

Three quantities in which we are particularly interested are the first moment of the pressure

$$<p> = i\omega\rho \sum_{n=1}^{M} \frac{<A_n(r)> \ \phi_n(z)}{\sqrt{k_n r}} e^{i(k_n r - \omega t)} \quad , \tag{5.19}$$

the second moment of the pressure

$$<|p(r,z,t)|^2> = \omega^2\rho^2 \sum_{n=1}^{M}\sum_{m=1}^{M} \frac{<A_n A_m^*> \ \phi_n(z)\phi_m(z)}{r\sqrt{k_n k_m}} e^{i(k_n - k_m)r} , \tag{5.20}$$

and the first moment of the acoustic Poynting vector

$$<\underline{j}> = \frac{\omega\rho}{2r} \sum_{n,m} \frac{1}{\sqrt{k_n k_m}} \Bigg\{ \hat{r}\,\phi_n(z)\phi_m(z) \left(k_m - \frac{1}{2r}\right) \mathrm{Re}\left[<A_n A_m^*> e^{i(k_n - k_m)r}\right]$$

$$+ \varepsilon\hat{r} \sum_{\ell} \mathrm{Re}\left[<A_n V_{m\ell}^* A_\ell^*> \ e^{i(k_n - k_m)r}\right] \phi_n(z)\phi_m(z)$$

$$- \hat{z}\phi_n(z) \frac{d\phi_m(z)}{dz} \ \mathrm{Im}\left[<A_n A_m^*> \ e^{i(k_n - k_m)r}\right] \Bigg\} \ . \tag{5.21}$$

Equations (5.19-5.21) are not exact expressions for these quantities since we have already dropped some higher-order terms in using Eq. (5.7). At any rate, we shall content ourselves with the lowest-order expressions for $\langle p \rangle$, $\langle |p|^2 \rangle$, and $\langle \underline{j} \rangle$. Thus, we drop the term proportional to ε in Eq. (5.21). In addition, since we are considering farfield propagation, we also drop the 1/2r term. Hence,

$$\langle \underline{j} \rangle \simeq \frac{\omega\rho}{2r} \Bigg\{ \sum_n \hat{r}\phi_n^2(z) \langle |A_n|^2 \rangle + {\sum_{n,m}}' \hat{r} \sqrt{\frac{k_m}{k_n}} \phi_m(z)\phi_n(z) \operatorname{Re}\left[\langle A_n A_m^* \rangle e^{i(k_n - k_m)r} \right] - \sum_{n,m} \hat{z} \frac{1}{\sqrt{k_n k_m}} \phi_n(z) \frac{d\phi_m(z)}{dz} \operatorname{Im}\left[\langle A_n A_m^* \rangle e^{i(k_n - k_m)r} \right] \Bigg\} . \tag{5.22}$$

Note that again $\langle \underline{j} \rangle$ is divided into self and interaction energy terms. The autocorrelation of a mode with itself represents the self-energy while interference effects are represented by the energy contained in correlations between the modes.

2. Calculation of the Moments

The scaling method used here to estimate the moments of $\underline{A}$, where $\underline{A} = (A_1, A_2, \ldots A_M)^T$, was introduced by Stratonovich.[53] The

literature extending this method to increasingly larger classes of problems is vast (see Ref. 50 and references within); the version of interest here was advanced by Papanicolaou and Kohler.[10] Although the work by Kohler and Papanicolaou[8] and Kohler[51] seem to be the only studies of the effects of internal waves on acoustic normal modes to employ rigorous scaling methods, these methods have been used in communications research for many years to describe wave propagation through waveguides with random inhomogeneities (e.g., Refs. 52 and 53).

The method entails scaling the range coordinate by the size of the fluctuations. In this scaled coordinate $\sigma=\varepsilon^2 r$, the effects of the random fluctuations are squeezed together enough that the acoustic field can be approximated by a Markov diffusion process (Appendix C) with an error of $O(\varepsilon)$ in the scaled coordinates, or $O(\varepsilon^3)$ in the original coordinate system. There are two great advantages to this. First, operational methods of treating Markov diffusion processes are available. Second, we have an estimate for the error in terms of its order in ε.

This approximation will hold at long ranges for small values of ε. Thus from Eq. (5.13)

$$\frac{d\underline{A}}{dr} = i\varepsilon \underline{\underline{V}}\,\underline{A} \quad , \tag{5.23}$$

where $\underline{\underline{V}}=\left(V_{nm}\right)$. As $\varepsilon\to 0$, $r\to\infty$ with $\sigma=\varepsilon^2 r$ fixed, moments of $\underline{A}$ converge to

corresponding moments of a Markov diffusion process $\underline{A}^0$. We assume that the sound speed fluctuations at two different ranges in the unscaled coordinate system become increasingly independent in a sufficiently strong sense (for a definition of "sufficiently strong" see Papanicolaou and Kohler[10]) as the range difference increases. We also demand that $0 \leq \sigma \leq \Sigma_0$, Σ_0 finite but arbitrarily large. The limitation of a finite Σ_0 has been lifted for stationary systems.[9] However, our system is not stationary in range, but in horizontal position. Thus, we are confined to the weaker results of Papanicolaou and Kohler.[10] This does not present a problem since the ocean isn't really infinite anyway.

The system (5.23) is completely analogous with the example in Remark 2 of Papanicolaou and Kohler;[10] if $\sigma_0 = \varepsilon^2 r_0$, $\gamma(\underline{A}^0, \underline{A}^{0*})$ is a function of the limiting Markov process $\underline{A}^0, \underline{A}^{0*}$, and

$$\Gamma\left(\underline{A}_0^0, \underline{A}_0^{0*}\right) = \int d\underline{A}^0 d\underline{A}^{0*} p(\underline{A}^0, \underline{A}^{0*}; \sigma | \underline{A}_0^0, \underline{A}_0^{0*}; \sigma_0) \, \gamma\left(\underline{A}^0, \underline{A}^{0*}\right) \quad , \qquad (5.24)$$

where $p(\underline{A}^0, \underline{A}^{0*}; \sigma | \underline{A}_0^0, \underline{A}_0^{0*}; \sigma_0)$ is the transition probability density of finding the system in state $\underline{A}^0, \underline{A}^{0*}$ at scaled range σ given initial conditions $\underline{A}^0, \underline{A}^{0*}(\sigma_0) = \underline{A}_0^0, \underline{A}_0^{0*}$, then $\Gamma(\underline{A}_0^0, \underline{A}_0^{0*})$ obeys a backward equation,

$$\frac{d\Gamma\left(\underline{A}_0^0, \underline{A}_0^{0*}\right)}{d\sigma_0} = \mathcal{L}\Gamma\left(\underline{A}_0^0, \underline{A}_0^{0*}\right) \quad , \qquad (5.25)$$

where

$$\mathcal{L} = \lim_{T\to\infty} \frac{1}{T}\int_{\sigma_0}^{\sigma_0+T} ds \int_{\sigma_0}^{s} dt \left\langle \left[\left(i\underline{\underline{V}}(t)\underline{A}_0^0, \frac{\partial}{\partial \underline{A}_0^0}\right) + \left(i\underline{\underline{V}}(t)\underline{A}_0^0, \frac{\partial}{\partial \underline{A}_0^0}\right)^*\right]\right.$$

$$\left. \times \left[\left(i\underline{\underline{V}}(s)\underline{A}_0^0, \frac{\partial}{\partial \underline{A}_0^0}\right) + \left(i\underline{\underline{V}}(s)\underline{A}_0^0, \frac{\partial}{\partial \underline{A}_0^0}\right)^*\right] \right\rangle \quad . \tag{5.26}$$

In Eq. (5.26), $\left(\frac{\partial}{\partial \underline{A}_0^0}\right)_n = \frac{\partial}{\partial A_{no}^0}$ and the inner product (A,B) is defined as

$$(A,B) = tr(AB^T) \quad . \tag{5.27}$$

Since limits have been placed on the amplitude of the acoustic signal (or else we can't start out with the linearized wave equation), we assume that $\gamma(\underline{A}^0,\underline{A}^{0*})$ vanishes for $\underline{A}^0,\underline{A}^{0*}$ larger than some bounds. If we also assume that first and second derivatives with respect to modal amplitude exist, then the transition probability density $p\left(\underline{A}^0,\underline{A}^{0*};\sigma|\underline{A}_0^0,\underline{A}_0^{0*};\sigma_0\right)$ obeys a forward, or Fokker-Planck, equation

$$\frac{dp\left(\underline{A}^0,\underline{A}^{0*};\sigma|\underline{A}_0^0,\underline{A}_0^{0*};\sigma_0\right)}{d\sigma} = \mathcal{L}^+ p\left(\underline{A}^0,\underline{A}^{0*};\sigma|\underline{A}_0^0,\underline{A}_0^{0*};\sigma_0\right) \quad , \tag{5.28}$$

where $\mathcal{L}^+$ is the formal adjoint of the operator defined in Eq. (5.26) and $p\left(\underline{A}^0,\underline{A}^{0*};\sigma_0|\underline{A}_0^0,\underline{A}_0^{0*};\sigma_0\right) = \delta\left(\underline{A}^0-\underline{A}_0^0\right)\delta\left(\underline{A}^{0*}-\underline{A}_0^{0*}\right)$. One additional

caveat: in order for the scaling approximation to hold, $\mathcal{L}^{+}$ must exist independently of σ_0. Papanicolaou and Kohler[10] assumed this in order to relax the condition that the fluctuating medium be stationary in range. There is no real way to know whether $\mathcal{L}^{+}$ is really independent of σ_0 for a nonstationary process without actually calculating it. This is not a problem when we neglect azimuthal coupling in the acoustic field, but we will need to remember this later when azimuthal coupling is taken into account.

Equation (5.28) is very useful in calculating propagation equations for the moments we need. Since we are approximating our real process by the limiting process, we will henceforth drop the superscript on $\underline{A}^{0}$. By multiplying Eq. (5.28) by A_k and integrating over $\underline{A}$ and $\underline{A}^{\star}$, we get a propagation equation for $\langle A_k \rangle$ (Appendix D):

$$\frac{d\langle A_k \rangle}{d\sigma} = -\frac{1}{\varepsilon^2}\sum_m (a_{mk} - i b_{mk}) \langle A_k \rangle \quad , \tag{5.29}$$

where

$$a_{mk} = \frac{8}{\pi^2}\left(\frac{\mu}{g}\right)^2 \langle \zeta^2(0) \rangle N_0 f \frac{\omega^4}{k_m k_k}\int_0^H dz \int_0^H dz' \rho(z)\rho(z')\phi_m(z)\phi_k(z)$$

$$\times \frac{\phi_m(z')\phi_k(z')}{\bar{c}^2(z)\bar{c}^2(z')} \frac{N^2(z)N^2(z')}{N(n)} \int_f^{N(n)} d\Omega \frac{(\Omega^2 - f^2)^{1/2}}{\Omega^3} a_\star$$

$$\times \int_0^{\pi/2} d\theta \frac{\cos\theta \cos\left[\frac{(k_m - k_k)}{\cos\theta} X(\Omega)\, \theta\right]}{\left[\alpha_\star^2 \cos^2\theta + (k_m - k_k)^2\right]} \tag{5.30}$$

and

$$b_{mk} = \frac{8}{\pi^2}\left(\frac{\mu}{g}\right)^2 \;<\zeta^2(o)>\; N_o f \frac{\omega^4}{k_m k_k}\int_o^H dz \int_o^H dz' \rho(z)\rho(z')\phi_m(z)\phi_k(z)$$

$$\times\; \phi_m(z')\phi_k(z') \; \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(n)} \int_f^{N(n)} d\Omega \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3}$$

$$\times \int_o^{\pi/2} d\theta \; \frac{\left[\alpha_* \cos\theta \sin\left[\frac{k_m-k_k}{\cos\theta}\mathcal{Z}X\right] + (k_m-k_k)\, e^{-\alpha_* X(\Omega)\mathcal{Z}}\right]}{\left(\alpha_*^2\cos^2\theta + (k_m-k_k)^2\right)} \quad . \tag{5.31}$$

Hence, since $\varepsilon^2 r = \sigma$, we find

$$<A_k(r)> \;=\; e^{-\sum_m (a_{mk}-ib_{mk})(r-r_o)} \;<A_k(r_o)> \quad . \tag{5.32}$$

The details of this calculation are given in Appendix D.

In a similar manner we derive propagation equations for the second moments:

$$\frac{d}{d\sigma} <A_k A_{k'}^*> \;=\; \frac{-1}{\varepsilon^2}\sum_m (a_{mk}+a_{mk'}+ib_{mk'}-ib_{mk}) \;<A_k A_{k'}^*> \qquad k\neq k' \quad , \tag{5.33}$$

so that

$$<A_k A_{k'}^*(r)> \;=\; e^{-\sum_m (a_{mk}+a_{mk'}+ib_{mk'}-ib_{mk})(r-r_o)} \;<A_k A_{k'}^*(r_o)> \qquad k\neq k' \tag{5.34}$$

and

$$\frac{d < |A_k(r)|^2 >}{dr} = 2 \sum_m a_{mk} \left(< |A_m(r)|^2 > - < |A_k(r)|^2 > \right) \quad . \tag{5.35}$$

Even with the oscillatory terms, $a_{mk} > 0$. Thus $< A_k(r) >$ and $< A_k A_{k'}(r) >$ oscillate with an exponentially decreasing envelope. The quantities $< |A_k|^2 >$ approach an equilibrium value

$$< |A_k|^2 > \rightarrow \frac{1}{M} \sum_{m=1}^{M} < |A_m(r_o)|^2 > \equiv \frac{E}{M} \quad . \tag{5.36}$$

Hence, knowledge of initial conditions is eventually wiped out as the energy is equipartitioned among the modes. From Eq. (5.9)

$$\frac{E}{M} = \frac{2\pi Q_o^2}{M} \rho^2(z_o) \sum_m \phi_m^2(z_o) \quad , \tag{5.37}$$

so that the equilibrium values for $< p >$, $< |p(r,z,t)|^2 >$, and $< \underline{j} >$ are

$$< p > \rightarrow 0 \quad , \tag{5.38}$$

$$< |p|^2 > \rightarrow \omega^2 \rho^2 \frac{E}{M} \sum_m \frac{\phi_m^2(z)}{k_m r} \quad , \tag{5.39}$$

and

$$< \underline{j} > \rightarrow \frac{\omega\rho}{2} \frac{E}{M} \sum_m \frac{\phi_m^2(z)}{r} \hat{r} \quad . \tag{5.40}$$

Chapter 6

In Chapter 5, we studied sound propagation where each depth mode was considered to have an azimuthally symmetric amplitude for each realization. This directly contradicts the assumption of azimuthal variations in the sound speed profile. In this chapter, we present a formalism for dealing with the azimuthal variations in the sound field.

A. The Coupled Mode Equations

We consider the same system as in the previous chapter, a flat slab of seawater of thickness H overlying a substrate. The velocity potential $\Psi(r,\varphi,z,t)$ satisfies the scalar wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi Q_0\delta(\underline{x}-\underline{x}_0)\ e^{-i\omega t} \quad . \qquad (6.1)$$

The sound speed profile is again considered "frozen" and we again expand Ψ in terms of the deterministic depth modes $\phi_n(z)$:

$$\Psi(r,\varphi,z,t) = \sum_{n=1}^{M} B_n(r,\varphi)\phi_n(z)\ \frac{e^{-i\omega t}}{\sqrt{r}} \quad . \qquad (6.2)$$

Thus

$$\frac{\partial^2 B_n(r,\varphi)}{\partial r^2} + \left(k_n^2 + \frac{1}{4r^2}\right) B_n(r,\varphi) + \frac{1}{r^2}\frac{\partial^2 B_n(r,\varphi)}{\partial \varphi^2}$$

$$= 2\omega^2 \sum_m \int_0^H dz\rho(z)\phi_n(z) \frac{\delta c}{\bar{c}^3} \phi_m(z) B_m(r,\varphi) \quad . \qquad (6.3)$$

We assume that internal wave effects have negligibly disturbed the acoustic field interior to a cylindrical boundary at $r_o=10/k_M$. Thus, Eq. (6.3) will describe the exterior problem with $B_n(r_o)$ obtained from Eq. (4.14) multiplied by $\sqrt{r_o}$. Accordingly, we drop the $1/4r^2$ and make the forward scattering approximation

$$B_n(r,\varphi) = \frac{1}{\sqrt{k_n}} B_n^+(r,\varphi)\, e^{ik_n r} \quad , \qquad (6.4)$$

so that

$$\frac{\partial B_n^+(r,\varphi)}{\partial r} + \frac{1}{2ik_n r^2}\frac{\partial^2 B_n^+(r,\varphi)}{\partial \varphi^2} = i\varepsilon \sum_m U_{nm} B_m^+(r,\varphi) \quad , \qquad (6.5)$$

where

$$U_{nm} = \frac{-\omega^2}{\varepsilon\sqrt{k_n k_m}} \int_0^H dz\rho(z)\phi_n(z) \frac{\delta c}{\bar{c}^3(z)} \phi_m(z)\, e^{i(k_m-k_n)r} \quad . \qquad (6.6)$$

Now we expand the forward scattered wave in terms of azimuthal modes

$$B_n^+(r,\varphi) = \sum_{q=-\infty}^{\infty} C_{nq}(r)\, e^{iq\varphi} \quad . \tag{6.7}$$

Hence

$$\frac{dC_{nq}}{dr} + \frac{iq^2}{2k_n r^2} C_{nq} = \frac{i\varepsilon}{2\pi} \sum_{mp} \int_0^{2\pi} d\varphi U_{nm}\, e^{i(p-q)\varphi}\, C_{mp} \quad . \tag{6.8}$$

Let $C_{nq} = A_{nq}\, e^{\frac{iq^2}{2k_n r}}$, then

$$\frac{dA_{nq}}{dr} = i\varepsilon \sum_{mp} V_{mpnq} A_{mp} \quad , \tag{6.9}$$

where

$$V_{mpnq} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi U_{nm}\, e^{i(p-q)\varphi} \exp\left[-i\left(\frac{q^2}{2k_n r} - \frac{p^2}{2k_m r}\right)\right] \quad , \tag{6.10}$$

and

$$A_{nq}(r_o) = \frac{B_n(r_o)}{\sqrt{k_n}}\, e^{ik_n r}\, \delta_{qo} \quad . \tag{6.11}$$

Since several changes of variable have been made, it is useful to exhibit these expressions in terms of basic functions,

$$B_n(r,\varphi) = \frac{1}{\sqrt{k_n}} \sum_q A_{nq} \exp\left[i\left(k_n r + \frac{q^2}{2k_n r} + q\varphi\right)\right] \quad , \tag{6.12}$$

$$V_{mpnq} = - \frac{\omega^2}{2\pi\varepsilon\sqrt{k_n k_m}} \int_0^{2\pi} d\varphi \int_0^H dz\rho(z)\phi_n(z) \frac{\delta c}{c^3} \phi_m(z)\ e^{i(k_m - k_n)r}$$

$$\times \exp\left[i\left(\frac{p^2}{2k_m r} - \frac{q^2}{2k_n r}\right)\right] e^{i(p-q)\varphi} \quad . \tag{6.13}$$

B. Calculation of Moments

As in the last chapter we will be interested in expressions for the average pressure,

$$\langle p \rangle = i\omega\rho \sum_{nq} \frac{\langle A_{nq}(r) \rangle}{\sqrt{k_n r}} \phi_n(z) \exp\left[i\left(k_n r + \frac{q^2}{2k_n r} + q\varphi - \omega t\right)\right] \ , \tag{6.14}$$

the second moment of the pressure,

$$\langle |p|^2 \rangle = \omega^2 \rho^2 \sum_{\substack{n,m \\ p,q}} \frac{\langle A_{nq} A^*_{mp} \rangle}{r\sqrt{k_n k_m}} \phi_n(z)\phi_m(z)$$

$$\times \exp\left\{i\left[(k_n - k_m)r + \left(\frac{q^2}{k_n} - \frac{p^2}{k_m}\right)\frac{1}{2r} + (q-p)\varphi\right]\right\} \ , \tag{6.15}$$

and the average energy flux,

$$\langle \underline{j} \rangle \approx \frac{\omega\rho}{2r} \sum_{\substack{nm\\pq}} \left\{ \frac{\hat{r}\phi_n(z)\phi_m(z)}{\sqrt{k_n k_m}} \left(k_m - \frac{p^2}{2k_m r^2} \right) + \hat{\varphi} \ \frac{p}{r}\ \phi_n(z)\phi_m(z) \right\}$$

$$\times \ \mathrm{Re} \left[\langle A_{nq} A^{*}_{mp} \rangle \exp\left\{ i \left[(k_n - k_m) r + \left(\frac{q^2}{k_n} - \frac{p^2}{k_m} \right) \frac{1}{2r} + (q-p)\varphi \right] \right\} \right]$$

$$- \frac{\omega\rho}{2r} \sum_{\substack{nm\\pq}} \ \hat{z}\phi_n(z) \ \frac{d\phi_m(z)}{dz} \ \mathrm{Im} \left[\langle A_{nq} A^{*}_{mp} \rangle \ \exp\left\{ i \left[(k_n - k_m) r \right.\right.\right.$$

$$\left.\left.\left. + \left(\frac{q^2}{k_n} - \frac{p^2}{k_m} \right) \frac{1}{2r} + (q-p)\varphi \right] \right\} \right] \quad . \qquad (6.16)$$

We will find that, as expected, a cylindrically symmetric source will require $q=0$ in Eq. (6.14) and though q need not be zero in Eqs. (6.15) and (6.16) it must be true that $q=p$.

Equation (6.9) for the matrix process $\underline{\underline{A}}=(A_{nq})$ is completely analogous to Eq. (5.23) for the vector process considered in Chapter 5. We shall place a restriction that q be less than some limit L where L is arbitrarily large. That is, we assume that we can always approximate a delta function by a function that is highly peaked at $\varphi=0$, the width of which is small compared with measured azimuthal variations in the velocity potential, and that we obtain

this function simply by adding together an adequate, but finite, number of azimuthal modes.

Given this restriction, we proceed in a manner analogous to the situation in Chapter 5. Rewriting Eq. (6.9),

$$\frac{d\underline{\underline{A}}}{dr} = i\varepsilon\, \underline{\underline{V}}\,\underline{\underline{A}} \quad , \tag{6.17}$$

where $\underline{\underline{A}}=(A_{nq})$ and $\underline{\underline{V}}=(V_{mpnq})$. With the scaling $\sigma=\varepsilon^2 r$, the processes $\underline{\underline{A}}\left(\frac{\sigma}{\varepsilon^2}\right)$ can be approximated by the Markov diffusion processes, $\underline{\underline{A}}^o(\sigma)$:

$$\frac{dp(\underline{\underline{A}}^o,\underline{\underline{A}}^{o*};\sigma|\underline{\underline{A}}^o,\underline{\underline{A}}_o^{o*};\sigma_o)}{d\sigma} = \mathcal{L}^{\dagger} p(\underline{\underline{A}}^o,\underline{\underline{A}}^{o*};\sigma\ |\underline{\underline{A}}_o^o,\underline{\underline{A}}_o^{o*};\sigma_o) \ , \tag{6.18}$$

where $\mathcal{L}^{\dagger}$ is the formal adjoint of

$$\mathcal{L} = \lim_{T\to\infty} \frac{1}{T}\int_{\sigma_o}^{\sigma_o+T} ds \int_{\sigma_o}^{s} dt \left< \left[\left(i\underline{\underline{V}}(t)\underline{\underline{A}}_o^o, \frac{\partial}{\partial \underline{\underline{A}}_o^o}\right) + \left(i\underline{\underline{V}}(t)\underline{\underline{A}}_o^o, \frac{\partial}{\partial \underline{\underline{A}}_o^o}\right)^*\right]\right.$$

$$\left. \times \left[\left(i\underline{\underline{V}}(s)\underline{\underline{A}}_o^o, \frac{\partial}{\partial \underline{\underline{A}}_o^o}\right) + \left(i\underline{\underline{V}}(s)\underline{\underline{A}}_o^o, \frac{\partial}{\partial \underline{\underline{A}}_o^o}\right)^*\right] \right> \quad . \tag{6.19}$$

The form of $\underline{\underline{V}}$ (Eq. 6.10) demands that our σ_o be nonzero (which it is) in order to calculate $\mathcal{L}^{\dagger}$ but we will find $\mathcal{L}^{\dagger}$ to be independent of σ_o when we're finished. In terms of the components of $\underline{\underline{V}}$ and $\underline{\underline{A}}$ (dropping the superscripts),

$$\mathcal{L}^{+} = \sum_{\substack{mpnq\\ m'p'n'q'}} \lim_{T\to\infty} \frac{1}{T} \int_{\sigma_0}^{\sigma_0+T} ds \int_{\sigma_0}^{s} dt$$

$$\left\{ \langle V^{(t)}_{mpnq} V^{(s)}_{m'p'n'q'} \rangle \left(\delta_{n'm}\delta_{pq'} \frac{\partial}{\partial A_{m'p'}} A_{nq} - \frac{\partial}{\partial A_{m'p'}} \frac{\partial}{\partial A_{mp}} A_{nq}A_{n'q'} \right) \right.$$

$$+ \langle V^{*(t)}_{mpnq} V^{*(s)}_{m'p'n'q'} \rangle \left(\delta_{n'm}\delta_{pq'} \frac{\partial}{\partial A^{*}_{m'p'}} - \frac{\partial}{\partial A^{*}_{m'p'}} \frac{\partial}{\partial A^{*}_{mp}} A^{*}_{nq}A^{*}_{n'q'} \right)$$

$$+ \langle V^{(t)}_{mpnq} V^{*(s)}_{m'p'n'q'} \rangle \frac{\partial}{\partial A^{*}_{m'p'}} \frac{\partial}{\partial A_{mp}} A^{*}_{n'q'}A_{nq}$$

$$\left. + \langle V^{*(t)}_{mpnq} V^{(s)}_{m'p'n'q'} \rangle \frac{\partial}{\partial A_{m'p'}} \frac{\partial}{\partial A^{*}_{mp}} A_{n'q'}A^{*}_{nq} \right\} . \qquad (6.20)$$

Each of the four terms will contain an expression of the form

$$\int_{-\infty}^{\infty} d\alpha \int_{0}^{2\pi} d\varphi \int_{0}^{2\pi} d\varphi' \int_{0}^{\pi/2} d\theta P(\alpha,\Omega;n) \cos(\alpha \not{z} X)\, e^{i\Delta q\varphi}\, e^{i\Delta q'\varphi'}$$

$$\times \left[e^{i\alpha[\sigma\cos(\varphi-\theta)-t\cos(\varphi'-\theta)]} + e^{i\alpha[\sigma\cos(\varphi+\theta)-t\cos(\varphi'+\theta)]} \right] . \qquad (6.21)$$

By transforming $\varphi \to \varphi+\theta$, $\varphi' \to \varphi'+\theta$ in the first integral and $\varphi \to \varphi-\theta$, $\varphi' \to \varphi'-\theta$ in the second integral, the expression (6.21) becomes

$$\int_{-\infty}^{\infty} d\alpha \int_{0}^{\pi/2} d\theta \int_{0}^{2\pi} d\varphi \int_{0}^{2\pi} d\varphi' P(\alpha,\Omega;n) \cos(\alpha \not{z} X)\, e^{i\alpha\sigma\cos\varphi}\, e^{-i\alpha t\cos\varphi'}$$

$$\times\left(e^{i\Delta q(\varphi+\theta)}\, e^{i\Delta q'(\varphi'+\theta)} + e^{i\Delta q(\varphi-\theta)}\, e^{i\Delta q'(\varphi-\theta)}\right)$$

$$= \frac{2\sin\left[\frac{\pi}{2}(\Delta q+\Delta q')\right]}{(\Delta q+\Delta q')} \int_{-\infty}^{\infty} d\alpha \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' P(\alpha,\Omega;n)\cos(\alpha \mathcal{Z} X)$$

$$\times\; e^{i\alpha\sigma\cos\varphi}\, e^{-i\alpha t\cos\varphi'}\, e^{i(\Delta q\varphi+\Delta q'\varphi')} \quad . \qquad (6.22)$$

If Δq and $\Delta q'$ are both even or both odd the integral is zero. Even more is required: because $P(\alpha,\Omega;n)$ is even in α and because the angle integrals go from 0 to 2π, the expression (6.22) is null unless $\Delta q=-\Delta q'$ (Appendix E). Hence,

$$\mathcal{L}^{\dagger} = \sum_{\substack{mpnq\\ m'p'n'q'}} \lim_{T\to\infty} \frac{1}{T} \int_{\sigma_0}^{\sigma_0+T} ds \int_{\sigma_0}^{s} dt$$

$$\Bigg\{ \langle V^{(t)}_{mpnq} V^{(s)}_{m'p'n'q'} \rangle \; \delta_{p,q-p'+q'} \left(\delta_{n'm}\delta_{pq'} \frac{\partial}{\partial A_{m'p'}} A_{nq} - \frac{\partial}{\partial A_{m'p'}} \frac{\partial}{\partial A_{mp}} A_{nq}A_{n'q'}\right)$$

$$+ \langle V^{*(t)}_{mpnq} V^{*(s)}_{m'p'n'q'} \rangle\, \delta_{p,q-p'+q'} \left(\delta_{n'm}\delta_{pq'} \frac{\partial}{\partial A^{*}_{m'p'}} A^{*}_{nq} - \frac{\partial}{\partial A^{*}_{m'p'}} \frac{\partial}{\partial A^{*}_{mp}} A^{*}_{nq}A^{*}_{n'q'}\right)$$

$$+ \langle V^{(t)}_{mpnq} V^{*(s)}_{m'p'n'q'} \rangle \; \delta_{p,q+p'-q'} \frac{\partial}{\partial A^{*}_{m'p'}} \frac{\partial}{\partial A_{mp}} A^{*}_{n'q'}A_{nq}$$

$$+ \langle V^{*(t)}_{mpnq} V^{(s)}_{m'p'n'q'} \rangle \; \delta_{p,q+p'-q'} \frac{\partial}{\partial A_{m'p'}} \frac{\partial}{\partial A^{*}_{mp}} A_{n'q'}A^{*}_{nq} \Bigg\} \quad . \qquad (6.23)$$

Another property common to the four terms in Eq. (6.18) is an expression of the form

$$\frac{1}{T}\int_{-\infty}^{\infty} d\alpha \int_{\sigma_0}^{\sigma_0+T} ds \int_{\sigma_0}^{s} dt\, P(\alpha,\Omega;n)\, \cos(\alpha \mathcal{J} X)\, e^{i\alpha(s\cos\varphi - t\cos\varphi')}$$

$$\times\, e^{i\Delta k_1 s}\, e^{i\Delta k_2 t}\, e^{(i/2s)\Delta Q_1^2}\, e^{(i/2t)\Delta Q_2^2} \quad , \qquad (6.24)$$

where, for example in the first term, $\Delta k_1 = (k_m - k_n)$, $\Delta k_2 = (k_{m'} - k_{n'})$, $\Delta Q_1^2 = \left(\frac{p^2}{k_m} - \frac{q^2}{k_n}\right)$, and $\Delta Q_2^2 = \left(\frac{p'^2}{k_{m'}} - \frac{q'^2}{k_{n'}}\right)$. Since $\left|\frac{\alpha_\star}{\alpha^2+\alpha_\star^2}\cos(\alpha \mathcal{J} X)\right| < 1/\alpha_\star$, ΔQ_1^2 and ΔQ_2^2 are constrained to be finite although arbitrarily large, σ_0 is finite (not zero) and less than T but otherwise arbitrary; and since no function of Ω multiplies T in the argument of an exponential, the expression in (6.24) can be written (Appendix F)

$$\frac{1}{T}\int_{-\infty}^{\infty} d\alpha \int_{\sigma_0}^{\sigma_0+T} ds \int_{\sigma_0}^{s} dt P(\alpha,\Omega;n)\, \cos(\alpha \mathcal{J} X)\, e^{i\alpha(s\cos\varphi - t\cos\varphi')}\, e^{i\Delta k_1 s}\, e^{i\Delta k_2 t}$$

$$+\, o\left(\frac{\ln T}{T}\right) + O\left(\frac{1}{T}\right) \quad . \qquad (6.25)$$

Since the terms $o(\ln T/T)$ and $O(1/T)$ will vanish in the limit $T\to\infty$, they are henceforth omitted.

It is now possible to calculate some useful moments. In the same manner as in the last chapter, a propagation equation for $\langle A_{kj} \rangle$ is obtained by multiplying Eq. (6.18) by A_{kj} and integrating by parts once. Thus

$$\frac{d\langle A_{kj}\rangle}{d\sigma} = -\sum_{mpn} \frac{\omega^4\left(\frac{\mu}{g}\right)^2}{\varepsilon^2\pi^4} \frac{\langle\zeta^2(o)\rangle N_0 f}{k_m\sqrt{k_n k_k}}$$

$$\times \lim_{T\uparrow\infty} \frac{1}{T} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \int_0^H dz \int_0^H dz' \phi_n(z)\phi_m(z)\phi_m(z')\phi_k(z')$$

$$\times \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(\Omega)} \rho(z)\rho(z') \quad \int_f^{N(\Omega)} d\Omega \int_{-\infty}^{\infty} d\alpha \, \frac{\alpha_\star}{(\alpha^2 + \alpha_\star^2)} \, \frac{(\Omega^2 - f^2)^{1/2}}{\Omega^3}$$

$$\times \int_{\sigma_0}^{\sigma_0+T} ds \int_{\sigma_0}^{s} dt \; e^{ip(\varphi-\varphi')} \; e^{-ij(\varphi-\varphi')} \; e^{i(k_m-k_n)s} \; e^{i(k_k-k_m)t}$$

$$\times \; e^{i\alpha(s\cos\varphi - t\cos\varphi')} \quad \langle A_{nj}\rangle \qquad . \qquad (6.26)$$

Now

$$\sum_{p=-L}^{L} e^{ip(\varphi-\varphi')} \simeq 2\pi\delta(\varphi-\varphi') \quad , \tag{6.27}$$

where this is approximate because of the truncated sum. Further manipulation yields

$$\frac{d\langle A_{kj}\rangle}{d\sigma} = -\sum_{mn} \frac{8\omega^4\left(\frac{\mu^2}{g^2}\right)\langle\zeta^2(o)\rangle N_o f}{\varepsilon^2\pi^3 k_m\sqrt{k_n k_k}}$$

$$\times \lim_{T\uparrow\infty} \frac{1}{T} \int_0^{\pi/2} d\varphi \int_0^H dz \int_0^H dz'\,\phi_n(z)\phi_m(z)\phi_m(z')\phi_k(z')$$

$$\times \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(n)}\,\rho(z)\rho(z') \int_f^{N(n)} d\Omega \int_{-\infty}^{\infty} d\alpha\, \frac{\alpha_\star}{(\alpha^2+\alpha_\star^2)}\, \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3}$$

$$\times \int_{\sigma_o}^{\sigma_o+T} ds \int_{\sigma_o}^{s} dt\; e^{i(k_m-k_n)s}\; e^{i(k_k-k_m)t}\; e^{i\alpha\cos\varphi(t-s)}\,\langle A_{nj}\rangle \quad . \tag{6.28}$$

Notice that the operator on $\langle A_{nj}\rangle$ does not depend on the indices of any of the azimuthal modes. In fact, this operator is precisely the one evaluated in Appendix D and

$$\frac{d\langle A_{kj}\rangle}{d\sigma} = -\frac{1}{\varepsilon^2}\sum_m (a_{mk}-ib_{mk})\,\langle A_{kj}\rangle \quad , \tag{6.29}$$

with a_{mk} and b_{mk} given by Eqs. (5.30) and (5.31). Thus

$$<A_{kj}> = e^{-\sum_m (a_{mk}-ib_{mk})(r-r_o)} <A_{kj}(r_o)> \quad . \qquad (6.30)$$

The derivation of Eq. (6.29) does not involve the initial conditions; the operator $\mathcal{L}^{\dagger}$ is purely a property of the medium and the acoustic frequency. Hence, Eq. (6.30) holds for directional sources as well as azimuthally symmetric sources; the first moment preserves the original directionality of the source. This is expected since the first moment does not give any information about energy transfer but merely serves as an indication of how quickly knowledge of the initial conditions is wiped out.

Information about the spatial distribution of energy is obtained from the second moments $<A_{nq}A^*_{n'q'}>$ (Appendix G). Here, it is necessary to return to the azimuthally symmetric source in order to make progress. We find first of all that the different azimuthal modes are uncorrelated to lowest order in ε; that is, we must have $q=q'$. The random pressure $p(r,\varphi,z,t)$ is thus stationary in the horizontal plane.

The interesting quantities involving second moments are

$$<|p|^2> = \sum_{n,n'} \frac{\omega^2\rho^2}{r} \frac{\phi_n(z)\phi_{n'}(z)}{\sqrt{k_n k_{n'}}} e^{i(k_n-k_{n'})r}$$

$$\times \sum_q <A_{nq}A^*_{n'q}> \exp\left[i\,\frac{q^2}{2r}\left(\frac{1}{k_n}-\frac{1}{k_{n'}}\right)\right] \quad , \qquad (6.31)$$

and

$$< \underline{j} > = \frac{\omega\rho}{2r} \sum_{\substack{nn'\\q}} \left\{ \left[\hat{r} \sqrt{\frac{k_{n'}}{k_n}}\, \phi_n(z)\phi_{n'}(z) \left(1 - \frac{q^2}{2k_{n'}^2 r^2} \right) \right.\right.$$

$$\left. + \hat{\varphi}\, \frac{q}{r}\, \frac{\phi_n(z)\phi_{n'}(z)}{\sqrt{k_n k_{n'}}} \right] \mathrm{Re} \left[< A_{nq} A^*_{n'q} > \exp \left\{ i \left[(k_n - k_{n'})r + \frac{q^2}{2r} \left(\frac{1}{k_n} - \frac{1}{k_{n'}} \right) \right] \right\} \right]$$

$$\left. - \hat{z}\, \frac{\phi_n \frac{d\phi_{n'}}{dz}}{\sqrt{k_n k_{n'}}}\, \mathrm{Im} \left[< A_{nq} A^*_{n'q} > \exp \left\{ i \left[(k_n - k_{n'})r + \frac{q^2}{2r} \left(\frac{1}{k_n} - \frac{1}{k_{n'}} \right) \right] \right\} \right] \right\} . \tag{6.32}$$

It is necessary here to make some estimate of our limit on q. The sound speed fluctuations have a correlation distance of about 10 km. Therefore, traversing an arc centered at the source, we expect changes of order one in the acoustic field over an arclength of about a correlation distance. We thus take $L = (2\pi r)/C_R$; our limit increases with range. Considering the radial term in $< \underline{j} >$,

$$\frac{L^2}{2k_{n'}^2 r^2} \sim \frac{2^{-2}}{(k_{n'} C_R)^2} \leq 2\times 10^{-3} \quad . \tag{6.33}$$

Similarly

$$\frac{L^2}{2r} \left(\frac{1}{k_n} - \frac{1}{k_{n'}} \right) \leq 2\times 10^{-3} (k_{n'} - k_n) r \quad . \tag{6.34}$$

Hence, we may drop the $\frac{\alpha^2}{2k_n^2r^2}$ term in $<\underline{j}>$ and can probably also drop the extra oscillatory terms (6.34) in $<|p|^2>$ and $<\underline{j}>$, depending on the size of $(k_{n'}-k_n)r$. In the estimates given above, $k_n \sim 10^{-2}\ m^{-1}$, about an order of magnitude smaller than what k_n would be at an acoustic frequency of 100 Hz.

The quantities $<A_{nq}A^*_{n'q}>$ satisfy the following equation:

$$\frac{d<A_{nq}A^*_{n'q}>}{d\sigma} = -\frac{1}{\varepsilon^2}\sum_m \left[(a_{mn'}+a_{mn}) + i(b_{mn'}-b_{mn})\right] <A_{nq}A^*_{n'q}>$$

$$+\frac{2}{\varepsilon^2}\sum_{mm'p} d^{pq}_{mnm'n'}\ \mathrm{Re}\ <A_{mp}A^*_{m'p}> \quad , \tag{6.35}$$

where a_{mn} and b_{mn} are given by Eqs. (5.30) and (5.31). The sums over m and m' are restricted such that $|k_n-k_m|\leq|k_{n'}-k_{m'}|$ and that (k_n-k_m) have the same sign as $(k_{n'}-k_{m'})$. The quantities $d^{pq}_{mnm'n'}$ are given by

$$d^{pq}_{mnm'n'} = \frac{16\omega^4}{\pi^3}\left(\frac{\mu}{g}\right)^2 N_0 f\ <\zeta^2(0)>\int_0^H dz\int_0^H dz'\ \frac{\phi_m(z)\phi_n(z')\phi_{m'}(z)\phi_{n'}(z')}{\sqrt{k_mk_{m'}k_nk_{n'}}}$$

$$\times\ \rho(z)\rho(z')\ \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(\omega)}\int_f^{N(\omega)} d\omega\ \frac{(\omega^2-f^2)^{1/2}}{\omega^3}\ \omega_*\int_0^{\pi/2} d\varphi\ \cos\varphi\cos\left[(q-p)\varphi\right]$$

$$\times\ \cos\left[(q-p)\cos^{-1}\left(\frac{k_n-k_m}{k_{n'}-k_{m'}}\cos\varphi\right)\right]\frac{\cos\left[\frac{k_{n'}-k_{m'}}{\cos\varphi}\ zX(\omega)\right]}{\left(\omega_*^2\cos^2\varphi+(k_{n'}-k_{m'})^2\right)} \ . \tag{6.36}$$

Note the similarity between $d^{pq}_{mnm'n'}$ and a_{mn} (Eq. 5.30). The excitation of the azimuthal modes by the internal wave field, bleeding energy out of the azimuthally symmetric mode, is explicit in Eq. (6.35). Since all the energy is initially in the q=0 mode, we see from the form of $d^{pq}_{mnm'n'}$ that $\langle A_{nq}A_{n'q}\rangle = \langle A_{n,-q}A^{*}_{n',-q}\rangle$ at all ranges. Therefore, $\langle \underline{j} \rangle$ does not have a component in the $\hat{\phi}$ direction.

The moments $\langle A_{nq}A_{n'q}\rangle$ are too many and too complicated to evaluate numerically. However, we obtain much information about energy flow without calculating each $\langle A_{nq}A_{n'q}\rangle$. Both $\langle |p|^2 \rangle$ and $\langle \underline{j} \rangle$ contain terms

$$w_n = \sum_q \langle |A_{nq}|^2 \rangle \quad , \tag{6.37}$$

which obey a propagation equation,

$$\frac{dw_n}{dr} = 2\sum_m a_{mn}(w_m - w_n) \quad . \tag{6.38}$$

This is precisely the same equation as that satisfied by $\langle |A_n|^2 \rangle$ (Eq. 5.35) for the case when azimuthal coupling was neglected. The self-energy is not affected by modal interference and so is simply the sum of the self-energies contained in all the azimuthal modes.

The interference terms are another story. For $n \neq n'$, summation over q in Eq. (6.35) yields

$$\frac{d}{d\sigma}\sum_q \langle A_{nq}A^*_{n'q}\rangle = -\frac{1}{\varepsilon^2}\sum_m \left[(a_{mn'}+a_{mn}) + i(b_{mn'}-b_{mn})\right]\sum_q \langle A_{nq}A^*_{n'q}\rangle \tag{6.39}$$

which is analogous to Eq. (5.33) for the azimuthally symmetric case. However, the expressions for $\langle |p|^2\rangle$ and $\langle \underline{j}\rangle$ contain expressions which oscillate more quickly than $\sum_q \langle A_{nq}A^*_{n'q}\rangle$:

$$\left|\sum_q \langle A_{nq}A^*_{n'q}\rangle \exp\left[\frac{iq^2}{2r}\left(\frac{1}{k_n}-\frac{1}{k_{n'}}\right)\right]\right| < \left|\sum_q \langle A_{nq}A^*_{n'q}\rangle\right| . \tag{6.40}$$

The effect of azimuthal coupling, then, enhances the decorrelation of the depth modes so that the interaction terms go more quickly to zero.

This enhancement is probably not very big. Changes in phase are much more rapid in range than in azimuth and a slight advancement or retardation of a depth mode as it travels in range will play much more havoc with acoustic interference than lateral deflections of energy. Still, the consideration of the extra dimension can only serve to increase randomness in the phases; this increase is represented by the additional oscillations in the correlations between different depth modes.

In view of the above considerations, the average energy flux is taken to be

$$
\begin{aligned}
<\underline{j}> \simeq \frac{\omega\rho}{2r} \Bigg\{ & \sum_n \hat{r}\phi_n^2(z) w_n(r) \\
& + \sum_{n,n'}{}' \hat{r} \sqrt{\frac{k_{n'}}{k_n}} \phi_n(z)\phi_{n'}(z) \sum_q \mathrm{Re}\Bigg(<A_{nq}A^*_{n'q}> \\
& \exp\left\{ i \left[(k_n - k_{n'})r + \frac{q^2}{2r}\left(\frac{1}{k_n} - \frac{1}{k_{n'}}\right)\right]\right\}\Bigg) \\
& - \sum_{n,n'}{}' \hat{z} \frac{\phi_n \frac{d\phi_{n'}}{dz}}{\sqrt{k_n k_{n'}}} \sum_q \mathrm{Im}\left(<A_{nq}A^*_{n'q}> \exp\left\{ i \left[(k_n - k_{n'})r + \frac{q^2}{2r}\left(\frac{1}{k_n} - \frac{1}{k_{n'}}\right)\right]\right\}\right)\Bigg\} .
\end{aligned}
$$

(6.41)

With the quantity E/M defined in Chapter 5 (Eq. 5.37), the asymptotic values for $<p(r,\varphi,z,t)>$, $<|p(r,\varphi,z,t)|^2>$, and $<\underline{j}>$ as $r\to\infty$ are

$$<p> \to 0 \quad , \tag{6.42}$$

$$<|p|^2> \to \omega^2\rho^2 \frac{E}{M} \sum_m \frac{\phi_m^2(z)}{k_m r} \quad , \tag{6.43}$$

and

$$<\underline{j}> \to \hat{r} \frac{\omega\rho}{2r} \frac{E}{M} \sum_m \phi_m^2(z) \quad , \tag{6.44}$$

as in the azimuthally symmetric case.

Chapter 7

In order to determine the rate of acoustic energy redistribution by the internal waves and the decay rate of the correlations between the different mode functions it is desirable to numerically calculate the scattering coefficients $a_{nn'}$. Because of the inequality (6.40), it is certainly necessary to know the decay rate of $\sum_q < A_{nq} A^*_{n'q} >$ since, if this quantity be negligible, the interference terms in $<|p|^2>$ and $<\underline{j}>$ may be neglected.

The scattering coefficients $a_{nn'}$ are given by

$$a_{nn'} = \frac{8\omega^4}{\pi^2}\left(\frac{\mu}{g}\right)^2 <\zeta^2(o)> \frac{N_o f}{k_n k_{n'}} \int_o^H dz \int_o^H dz' \rho(z)\rho(z')$$

$$\times \frac{\phi_n(z)\phi_{n'}(z)\phi_n(z')\phi_{n'}(z')}{\bar{c}^2(z)\bar{c}^2(z')N(\eta)} N^2(z)N^2(z') \int_f^{N(\eta)} d\Omega \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3} \alpha_\star$$

$$\times \int_o^{\pi/2} d\varphi \frac{\cos\varphi \cos\left(\frac{\Delta k \mathfrak{z} X(\Omega)}{\cos\varphi}\right)}{(\alpha_\star^2\cos^2\varphi+\Delta k^2)} \quad , \qquad (7.1)$$

where

$$\Delta k = |k_n - k_{n'}| \quad , \qquad (7.2)$$

$$\mathfrak{z} = |z-z'| \quad , \qquad (7.3)$$

$$\eta = \frac{1}{2}(z+z') \quad , \tag{7.4}$$

$$\alpha_* = t(\Omega^2-f^2)^{1/2} \quad , \tag{7.5}$$

and

$$X(\Omega) = \sqrt{\frac{N^2 - \Omega^2}{\Omega^2 - f^2}} \quad , \tag{7.6}$$

The Ω integration can be transformed by a change of variable,

$$\int_f^{N(\eta)} d\Omega \, \frac{\alpha_* (\Omega^2-f^2)^{1/2}}{\Omega^3} \, \frac{\cos\left(\frac{\Delta k \mathfrak{z} X(\Omega)}{\cos\varphi}\right)}{(\alpha_*^2 \cos^2\varphi + \Delta k^2)} \rightarrow \frac{t(N^2-f^2)^2}{\Delta k^2 f^4} \int_0^\infty dX \, \frac{X \cos(GX)}{\left(X^2+K^2\right)^2 (X^2+Q^2)} \quad , \tag{7.7}$$

where

$$G = \frac{\Delta k \mathfrak{z}}{\cos\varphi} \quad , \tag{7.8}$$

$$K = N(\eta)/f \quad , \tag{7.9}$$

$$Q^2 = \frac{t^2(N^2-f^2)\cos^2\varphi}{\Delta k^2} + 1 \quad . \tag{7.10}$$

For $\Delta k > ft$, we can separate the denominator of the integrand in (7.7) into partial fractions and evaluate the X integral exactly. For $\Delta k < ft$, the partial fraction expansion will introduce second-order

poles at ft cos $\varphi = \Delta k$ and then the principal value of the φ integral will not exist. It is certainly true that the expression (7.1) for $a_{nn'}$ is valid uniformly in latitude and bandwidth parameter t, but one must be careful that methods chosen to evaluate the integrals be valid.

Evaluation of $a_{nn'}$ is greatly simplified by assuming $\Delta k > ft$. This limits the latitudes at which the following expressions apply, but this does not prevent a useful computation. For t = 1.4 sec/m, consistent with $j_{\star}=3$ in Desaubies'[36] formalism, and a latitude of 30^0, ft = 1.02×10^{-4} rad/m. At any rate, the expression (7.1) is as refined as it can get without introducing some approximations. Numerical evaluation of the quadruple integral for each pair of depth modes would take a prohibitive amount of computer time, even with Monte-Carlo techniques of integration. Since we do not expect the qualitative behavior of the acoustic field to vary drastically with latitude, we solve the problem for $\Delta k > ft$.

Performing the internal wave frequency integral, we find (with N>>f; Appendix H)

$$a_{nn'} = \frac{4\omega^4}{\pi^2}\left(\frac{\mu}{g}\right)^2 \langle \zeta^2(o) \rangle \frac{N_o ft}{\Delta k^2 k_n k_{n'}} \int_o^H dz \int_o^H dz' \rho(z)\rho(z')\phi_n(z)\phi_{n'}(z)$$

$$\times \frac{\phi_n(z')\phi_{n'}(z')N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(n)} \left\{ \frac{-\Delta k^2}{ft\sqrt{\Delta k^2 - f^2t^2}} \tan^{-1}\left(\frac{ft}{\sqrt{\Delta k^2 - f^2t^2}}\right)\right.$$

$$+\int_0^{\pi/2} d\varphi \cos\varphi \left(\frac{GK}{2\left(1 - \frac{f^2t^2}{\Delta k^2}\cos^2\varphi\right)} \left[e^{-GK}\,\mathrm{Ei}(GK) - e^{GK}\,\mathrm{Ei}(-GK)\right]\right.$$

$$+\left(1 - \frac{f^2t^2}{\Delta k^2}\cos^2\varphi\right)^{-2}\left[e^{-GK}\,\mathrm{Ei}(GK) + e^{GK}\,\mathrm{Ei}(-GK)\right]$$

$$\left.\left.-\left(1 - \frac{f^2t^2}{\Delta k^2}\cos^2\varphi\right)^{-2}\left[e^{-GQ}\,\mathrm{Ei}(GQ) + e^{GQ}\,\mathrm{Ei}(-GQ)\right]\right)\right\} \quad , \qquad (7.11)$$

where Ei(x) is the exponential integral.

Because the exponential integral diverges near $\mathcal{z}=0$, it is interesting to note that as $G\to 0$ the terms in curly brackets become

$$\frac{-\Delta k^2}{ft\sqrt{\Delta k^2 - f^2t^2}}\tan^{-1}\left(\frac{ft}{\sqrt{\Delta k^2 - f^2t^2}}\right) + 2\int_0^{\pi/2} d\varphi \cos\varphi \frac{(\ln K - \ln Q)}{\left(1 - \frac{t^2f^2}{\Delta k^2}\cos^2\varphi\right)^2} \quad , \qquad (7.12)$$

a result which can also be obtained by integrating the integral (7.7) with G=0 (Appendix H).

The expression (7.11) for $a_{nn'}$ looks much more daunting than it actually is. The φ integrand is actually very smooth and well-behaved except near $\mathcal{z}=0$. We can eliminate much repetition in the numerical evaluation of $a_{nn'}$ by noting that the expression (7.11) is of the form

$$a_{nn'} = \int_0^H dz \int_0^H dz'\, h(z)h(z')g(\mathcal{z},\eta) \quad , \qquad (7.13)$$

which is equivalent to

$$a_{nn'} = 2\left[\int_0^{H/2} d\eta \int_0^{2\eta} d\mathcal{z} + \int_{H/2}^{H} d\eta \int_0^{2(H-\eta)} d\mathcal{z}\right] h(\eta-\mathcal{z}/2)h(\eta+\mathcal{z}/2)g(\mathcal{z},\eta) . \tag{7.14}$$

After one more transformation $\mathcal{z}/2 \to \Theta$,

$$a_{nn'} = 4\left[\int_0^{H/2} d\eta \int_0^{\eta} d\Theta + \int_{H/2}^{H} d\eta \int_0^{(H-\eta)} d\Theta\right] h(\eta-\Theta)h(\eta+\Theta)g(2\Theta,\eta) . \tag{7.15}$$

The scattering coefficients $a_{nn'}$ were calculated for n=1, n'=2-6 at acoustic frequencies of 100 Hz and 50 Hz. We assume an exponentially decreasing Brunt-Väisälä frequency

$$N(z) = N_o\, e^{-z/B} , \tag{7.16}$$

where N_o = 0.00524 rad/sec and B = 1300 m. Correspondingly, a Munk profile was chosen for the deterministic sound speed

$$\overline{c}(z) = c_o\left[1 + \varepsilon\left(\Delta+e^{-\Delta}-1\right)\right] , \tag{7.17}$$

where c_o = 1500 m/sec, $\varepsilon=0.737\times10^{-2}$,

$$\Delta = \frac{2}{B}\,(z-z_o) , \tag{7.18}$$

and z_o = 1300 m. The inertial frequency was taken to be 0.727×10^{-4} rad/sec (30^o latitude). The bandwidth parameter t was chosen to be

1.4 sec/m, consistent with $j_{\star}=3$ in Desaubies'[36] notation. Finally, in order to compare our numerical results with those of Dozier and Tappert,[7] we choose the strength of the fluctuations such that

$$\left(\frac{\mu}{g}\right)^2 \langle \zeta^2(o) \rangle N_0 = 2.12\ s^3 \quad . \tag{7.19}$$

The acoustic wave numbers k_n and the corresponding depth modes ϕ_n were calculated for a rigid bottom using the normal mode model NEMESIS[54] on a CYBER 171 computer. Briefly, boundary conditions were set at z=0 and z = H = 4000 m. Given an eigenvalue estimate, Numerov's method is used to integrate the differential equation (5.2) up from z=H to a match point and down to that match point from the surface. The difference in these two solutions at the match point, if not within the required tolerance, is then used to determine the next eigenvalue estimate. Polynomial expressions involving previously calculated eigenvalues are used as estimates for the next eigenvalue so that convergence is usually achieved in five or six iterations. Each eigenfunction ϕ_n was evaluated every 5 m so that there were 801 mesh points in the 4000 m of water.

Because the φ integrand is so smooth and since the φ integral must be evaluated for each (η, Ω) pair, the integral over azimuth was found to be evaluated most efficiently using a 2^I point Gaussian tensor product formula,[55] where I varies from 1 to 8 until

the required accuracy is achieved. The IMSL routine DMLIN[55] was used to effect this quadrature.

The depth integrals were evaluated by a two-dimensional trapezoidal rule which turned out to be surprisingly accurate. We estimate the calculated values of $a_{nn'}$ to be within 5% and probably [illegible] 3%.

The values of $a_{nn'}$ calculated in this study are presented in Table 7.1 along with corresponding values calculated by Dozier and Tappert.[7] The two treatments are seen to agree fairly closely.

TABLE 7.1

Calculation of $a_{1n'}$

100 Hz

n'	$a_{1n'}$(Penland)	$a_{1n'}$(Dozier-Tappert)
2	.704E-6	.945E-6
3	.140E-6	.208E-6
4	.525E-7	.760E-7
5	.264E-7	.321E-7
6	.155E-7	.147E-7

50 Hz

n'	$a_{1n'}$(Penland)	$a_{1n'}$(Dozier-Tappert)
2	.246E-6	.294E-6
3	.548E-7	.710E-7
4	.210E-7	.283E-7
5	.104E-7	.140E-7
6	.603E-8	.775E-8

Note: Values from Dozier and Tappert[7] have been divided by 2 to make consistent with ours their definition of $a_{nn'}$.

Chapter 8

Mode coupling induced by internal waves redistributes the energy contained in an acoustic mode field. Energy confined to the SOFAR channel will be spread over a much larger depth at long ranges.

This scattering causes the behavior of the sound field at long ranges to become independent of source depth. A limit on the depth information content of a cw signal which has propagated over a long distance has therefore been established.

This study presents a theory which describes this redistribution of energy. Path integral descriptions employed by Flatté et al.[4] can be used to describe acoustic propagation through a random medium at frequencies high enough to make normal mode techniques invalid or, especially, intractable. However, unlike these path integral techniques, this normal mode description is valid at low frequencies ($\nu \lesssim 100$ Hz) and does not fall prey to problems of caustics or to those connected with describing the acoustic field between the saturated and unsaturated regions.

The scaling techniques introduced by Kohler and Papanicolaou[8] have been extended to include azimuthal fluctuations in the sound speed field. Two cases are considered. In one case, the azimuthal fluctuations in the acoustical field were ignored while

retaining ıch fluctuations in the sound speed. Thi
contradiction. This case was studied to see if the resonance scattering in the Dozier-Tappert[6,7] theory without manipulating the spectrum in such a manner a tal isotropy, an average property, on each realizati speed. Also, we wished to investigate the necessity assumption, i.e., $\langle A_n A^*_{n'} \rangle = 0$ for $n \neq n'$, in obtaining Eqs. (5.35). In addition, many concepts are elucida would be obscured by the more complicated case.

In the other case, the azimuthal fluctuati acoustic field are taken into account. The results study are then compared with those of the previous, symmetric situation.

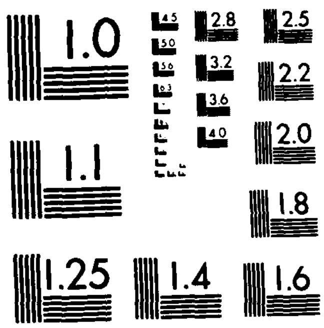
The azimuthally symmetric case is presente Chapter 5. The contradiction inherent in a sit.atic variations are permitted in the sound speed and not field is explicit in the fact that the random pressu obey an azimuthally symmetric Helmholtz equation wh somehow "localized" (see the sentence immediately be Appendix D).

Contradictions notwithstanding, equations amplitude $\langle A_n \rangle$, for the autocorrelation function mode amplitude and for the correlations $\langle A_n A^*_{n'} \rangle$ b ferent mode amplitudes (see Eqs. 5.6, 5.10-5.13) we

AD-A150 235 THE EFFECTS OF INTERNAL WAVES ON ACOUSTIC NORMAL MODES 2/2
(U) TEXAS UNIV AT AUSTIN APPLIED RESEARCH LABS
M C PENLAND DEC 84 ARL-TR-84-14 N00014-80-C-0490

UNCLASSIFIED F/G 20/1 NL

END



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

The function $\langle A_n \rangle$ oscillates with an exponentially decreasing envelope (Eq. 5.32). Hence, the mean acoustic pressure becomes small at long ranges more quickly than would be implied by simple cylindrical spreading.

The correlation functions $\langle A_n A_{n'}^* \rangle$ also oscillate with an exponentially decreasing envelope (Eq. 5.34). This envelope decreases more rapidly than that of the first moment; the e-folding range is about half that of $\langle A_n \rangle$.

The autocorrelation function $\langle |A_n|^2 \rangle$ of each mode is coupled with those of all the other modes (Eq. 5.35). At long ranges all of the functions $\langle |A_n|^2 \rangle$ approach the same equilibrium value; all depth information about the source is therefore eventually lost.

Because of the behavior of $\langle |A_n|^2 \rangle$ and $\langle A_n A_{n'}^* \rangle$, the second moment of the acoustic pressure $\langle |p|^2 \rangle$ asymptotically approaches an incoherent sum of contributions from the the normal modes (Eq. 5.39). The vertical components of the average energy flux $\langle \underline{j} \rangle$ decrease with range so that the average energy flux at long ranges is essentially radial (Eq. 5.40).

Even without the assumption of random phases, the form of the coupled equations for $\langle |A_n|^2 \rangle$ is the same as that derived by Dozier and Tappert.[6] Of course, Kohler and Papanicolaou[8] obtained the same form without assuming random phases so this result is not at all surprising. The form of Eqs. (5.32), (5.34), and (5.35) is a consequence of approximating the system by a Markov diffusion process. At any rate, we can check the consistency of assuming random phases

with the Dozier-Tappert results since the exponentially decreasing envelope of $\langle A_n A^*_{n'} \rangle$ contains the same coefficients a_{mn} as in the coupled equations for $\langle |A_n|^2 \rangle$.

Using the values for these quantities given in Table 1 of Ref. 7 for a 100 Hz signal, we calculate $\sum_m (a_{m1}+a_{m2}) = 3.8 \times 10^{-6}\ m^{-1}$, corresponding to an e-folding length of 265.5 km, or about 25 correlation lengths. Of course, they only tabulated the results for ten modes whereas there are generally hundreds of modes in the deep (4000 m) ocean. Nevertheless, the scattering coefficients between modes 1 and m decrease with the difference in mode number somewhat more rapidly than $|1-m|^{-2}$. Thus, if we assume a very large number of modes, we can overestimate $\sum_m (a_{m1}+a_{m2}) = 4.3 \times 10^{-6}\ m^{-1}$, corresponding to an e-folding length of 231.2 km, or about 21 correlation lengths. On the other hand, the gravest acoustic mode has only one nearest neighbor, so we would expect it to be highly correlated with mode 2. A similar calculation of $\sum_m (a_{m5}+a_{m6})$ at 100 Hz gives an e-folding length of about 67.2 km, or about 6.5 correlation lengths. Hence, we expect the random phase approximation in the Dozier-Tappert theory to be most valid for mode numbers of middle range. Lower acoustic frequencies will have longer e-folding lengths, varying roughly as ω^2.

The calculation of the e-folding lengths was performed with scattering coefficients calculated by Dozier and Tappert[7] using the 1975 Garrett-Munk spectrum.[33] Their spectrum differed from the one used in this study in that their horizontal wave number spectrum fell

off more rapidly with α than the more recently derived spectrum used in the analysis presented here. The 1975 Garrett-Munk spectrum[33] was revised with subsequent experiments[32,34,35] and we have used the revised version. On the other hand, their bandwidth parameter t is about twice as large as that indicated by these experiments and we did not obtain scattering coefficients drastically different from their values (Table 7.1).

From the form of the scattering coefficients (Eqs. 5.30 and 5.31) it is clear that they represent contributions from the internal wave field having a component of horizontal wave number in the radial direction equal to the difference in acoustic horizontal wave numbers. Dozier and Tappert[6] interpreted their scattering coefficients as representing contributions from the internal wave field with horizontal wave numbers equal to the difference of the acoustic horizontal wave numbers. However, since they integrated their two-dimensional spectrum over transverse wave number, we may loosely identify the two treatments as having consistent interpretations in this respect.

At any rate, one must always remember that this "resonance coupling" is derived from the fact that both the horizontal wave number spectrum used here and the modified spectrum used by Dozier and Tappert possess poles of no higher order than one on the imaginary axis. These poles did not contribute to the coupling coefficients a_{mn} although the wavelength of the oscillation of $\langle A_n A_{n'}^* \rangle$ was indeed affected by their presence in the spectrum (Eq. 5.31). A second-order

pole in the horizontal wave number spectrum would contribute additional terms to a_{mn} and a third-order or higher-order pole could call into question the approximation of the system $\{A_n\}$ by a Markov diffusion process.

Because the system $\{A_n\}$ would be identically constant in a noise-free environment and since the sound speed fluctuations are small, continuous, and have a reasonable correlation length, it is reasonable to expect a Markov diffusion process to approximate $\underline{A}$ with proper scaling. The assumption that such a process exists itself places some restrictions on the form of the horizontal wave number spectrum, but there is still a good deal of freedom in choosing an expression to fit the measured data. Hence, although the expressions for a_{mn} and b_{mn} can be trusted to give accurate numerical results, we should remember that the "resonance coupling" might have been modified had we used another empirically derived spectrum which also fits well the measured internal wave data but has different analytic properties.

With a thorough understanding of the azimuthally symmetric case, we now turn to the more realistic situation where azimuthal fluctuations in the acoustic field are taken into account. Although it is no longer necessary to "localize" the Helmholtz equation, we pay for this consistency with a large increase in complexity. The details of this calculation are presented in Chapter 6.

Many of the results found in the azimuthally symmetric case have direct analogies in the more complicated study. Each mode amplitude is a function of azimuth for each realization; each is

therefore expressed in terms of a set of azimuthal modes. The average pressure $\langle p \rangle$ due to a point source is precisely the same as in the azimuthally symmetric case. However, we have more information: the calculation of $\langle p \rangle$ does not depend on an azimuthally symmetric source. We find that $\langle p \rangle$ preserves the directionality of the acoustic source and that each mode amplitude oscillates in range with an exponentially decreasing envelope.

When the source is an azimuthally symmetric point source it is found that the acoustic pressure is a stationary random process in the horizontal plane. The autocorrelation function for the depth modes have exactly the same behavior as in the azimuthally symmetric case provided the range functions $\sum_q \langle |A_{nq}|^2 \rangle$ are interpreted as a sum over all the azimuthal modes; energy is eventually equipartitioned among all of the depth modes (compare Eqs. 5.35 and 6.38). As far as the correlations between different depth modes are concerned, the quantity $\sum_q \langle A_{nq} A^*_{n'q} \rangle$ behaves as does $\langle A_n A^*_{n'} \rangle$ in the azimuthally symmetric case; it oscillates in range with an exponentially decreasing envelope (Eq. 6.39). However, the interference terms in $\langle |p|^2 \rangle$ and $\langle \underline{j} \rangle$ oscillate in q^2; the quantity $|\sum_q \langle A_{nq} A^*_{n'q} \rangle|$ must be regarded as an upper bound to these interference terms. Hence, azimuthal scattering of energy tends to slightly enhance phase randomization so that the interference terms are encouraged to decay.

Therefore, the asymptotic behavior of $\langle p \rangle$, $\langle |p|^2 \rangle$, and $\langle \underline{j} \rangle$ is the same as in the case where azimuthal fluctuations in the acoustic field are not considered. However, this asymptotic behavior

may be obtained more quickly than would be predicted by the simpler calculation, though this difference in range is probably not significant. Criteria for determining the significance of the randomization enhancement are given in Chapter 6.

The result that energy is eventually equipartitioned among all the depth modes is dependent upon the neglect of the continuous spectrum. Unfortunately, the reasons usually cited as justification for this neglect are precisely those mandating consideration of energy coupling into this radiation spectrum: this continuum region is attenuated far more rapidly than are the discrete depth modes. Analytical work by Kohler and Papanicolaou[8] indicates that equipartition will not be attained if the continuum is included. Clearly, further work is needed to establish the rate of such coupling via the internal wave field.

Since the measure of volume absorption in seawater involves propagation of an acoustic signal over very long ranges (10^2-10^3 km), internal waves will scatter energy out of the paths connecting a point source with a point receiver. This may result in an anomalously high volume "absorption." The theory presented here should be modified to include the effects of absorption and a numerical calculation which includes absorption should be performed.

Finally, one could apply the methods of Chapter 6 to modify Kohler's[51] results for pulse propagation through a random environment

for the special case that the random sound speed inhomogeneities be due to internal waves. The only differences one would expect between the Kohler theory as it stands and the modified Kohler theory would be in the interference terms, which would be small at very long ranges, and in the interpretation of the depth mode autocorrelation functions, each of which would now be a sum of azimuthal modes.

APPENDIX A

Munk's[19] derivation of the mean sound speed profile $\overline{c}(z)$ is presented here.

Neglecting second-order terms, the fractional sound speed gradient can be written

$$\overline{c}^{-1} \frac{\partial \overline{c}}{\partial z} = \alpha \frac{\partial T}{\partial z} + \beta \frac{\partial S}{\partial z} + \gamma \frac{\partial p}{\partial z} \quad . \tag{A.1}$$

Here, T is temperature, S is salinity, p is pressure, and α, β, γ are constants. Similarly, the fractional density gradient is

$$\rho^{-1} \left(\frac{\partial \rho}{\partial z}\right)_P = -a\left(\frac{\partial T}{\partial z}\right)_P + b \frac{\partial S}{\partial z} \quad , \tag{A.2}$$

where a is the coefficient of thermal expansion and b is the coefficient of saline contraction. The subscript P denotes "potential gradient", as explained in Chapter 2. Only the potential density gradient contributes to the stability of the water column; only the potential sound speed gradient contributes to sound fluctuations associated with internal waves. It is therefore convenient to introduce the quantity γ_A, the fractional sound speed gradient in an isentropic, isohaline ocean.

$$\gamma_A \equiv \alpha \left(\frac{\partial T}{\partial z}\right)_A + \gamma \frac{\partial p}{\partial z} \quad . \tag{A.3}$$

Values for coefficients typical of conditions in the ocean are given in Table A.1. However, it should be noted that a may vary by as much as

TABLE A.1

Coefficients used in Derivation of Munk Profile

Values of $10^6 a$ in $(°C)^{-1}$. at stated temperatures and depths

Depth (m)	T(°C) 0	5	10	15	20	25
0	51	107	158	204	245	283
1000	76	127	173	216	254	290
2000	100	146	189	227	263	296
3000	122	164	203	238	271	302
4000	143	181	217	249	279	308
5000	161	197	229	258	286	314

$\alpha = 3.16 \times 10^{-3} (°C)^{-1}$, $a \approx 0.13 \times 10^{-3} (°C)^{-1}$,

$b = 0.80 \times 10^{-3} (‰)^{-1}$, $\beta = 0.96 \times 10^{-3} (‰)^{-1}$,

$\gamma = 1.11 \times 10^{-2}\ km^{-1}$.

50% between the sea surface and the ocean floor. The value of a given at the bottom of Table A.1 represents a channel value for a.

The relative contributions of salt and (potential) temperature to the stability of the ocean is introduced through the "Turner number" r,[56]

$$r \equiv \frac{b\left(\frac{\partial S}{\partial z}\right)}{a\left(\frac{\partial T}{\partial z}\right)_p} \quad . \tag{A.4}$$

We are now in a position to combine Eqs. (A.1-A.4) to obtain an expression for $\bar{c}^{-1}\frac{\partial \bar{c}}{\partial z}$. Using the Brunt-Väisälä frequency,

$$N^2(z) = \frac{g}{\rho}\left(\frac{\partial \rho}{\partial z}\right)_p \quad , \tag{A.5}$$

we find

$$\bar{c}^{-1}\frac{\partial \bar{c}}{\partial z} = -\left(\frac{\mu}{g}\right)N^2(z) + \gamma_A \quad , \tag{A.6}$$

where

$$\mu \equiv \frac{\alpha}{a}\left[\frac{1 + \left(\frac{a\beta}{\alpha b}\right) r}{1 - r}\right] \quad . \tag{A.7}$$

The sound speed channel axis is located where $\frac{\partial \bar{c}}{\partial z}$ vanishes, that is, where

$$N(z_o) = \left(\frac{\gamma_A g}{\mu}\right)^{1/2} \quad . \tag{A.8}$$

For exponentially stratified oceans,

$$N(z) = N_o e^{-z/B} \tag{A.9}$$

and

$$z_o = \frac{1}{2} B \ln\left(\frac{\mu N_o^2}{\gamma_A g}\right) \quad . \tag{A.10}$$

For a "surface extrapolated" value of $N(0) = N_o = 3\ h^{-1}$, $B = 1.3$ km and $\gamma_A = 1.14 \times 10^{-2}\ \text{km}^{-1}$ (1 km of seawater exerts 99 bars of pressure),

$$z_o = 1.16\ \text{km} + \frac{1}{2} B \ln\left[\frac{1 + \frac{a\beta}{\alpha b} r}{1 - r}\right] \quad , \tag{A.11}$$

which is generally about the same size as B.

If we define the dimensionless variables,

$$\Delta = 2\ \frac{(z-z_o)}{B} \tag{A.12}$$

and

$$\varepsilon = \frac{1}{2} B\gamma_A \quad , \tag{A.13}$$

the Munk sound speed profile is found to be

$$\bar{c}(z) = \bar{c}(z_o) \exp\left[\varepsilon(\Delta+e^{-\Delta}-1)\right] \quad , \tag{A.14}$$

or, for small ε,

$$\bar{c}(z) = \bar{c}(z_o)\left[1 + \varepsilon(\Delta+e^{-\Delta}-1) + O(\varepsilon^2\Delta^4)\right] \quad . \tag{A.15}$$

APPENDIX B

We write our three-dimensional particle velocity due to internal waves $\underline{u}_i=(u,v,w)$. In keeping with standard notation,[4,25] u is not the magnitude of $\underline{u}_i$, but rather its x component. Thus

$$\frac{\partial u}{\partial t} + (\underline{f}\times\underline{u}_i)_x + \frac{1}{\rho_o}\frac{\partial p_i}{\partial x} = 0 \quad , \tag{B.1}$$

$$\frac{\partial v}{\partial t} + (\underline{f}\times\underline{u}_i)_y + \frac{1}{\rho_o}\frac{\partial p_i}{\partial y} = 0 \quad , \tag{B.2}$$

$$\frac{\partial w}{\partial t} + (\underline{f}\times\underline{u}_i)_z + \frac{1}{\rho_o}\frac{\partial p_i}{\partial z} - \rho_i\frac{g}{\rho_o} = 0 \quad , \tag{B.3}$$

$$\frac{\partial \rho_i}{\partial t} + w\frac{\partial \rho_o}{\partial z} = 0 \quad , \tag{B.4}$$

$$\nabla\cdot\underline{u}_i = 0 \quad . \tag{B.5}$$

From (B.2) and (B.3),

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial}{\partial z}\frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial y^2}(\underline{f}\times\underline{u}_i)_z - \frac{\partial}{\partial z}\frac{\partial}{\partial y}(\underline{f}\times\underline{u}_i)_y$$
$$+ \frac{1}{\rho_o}\frac{\partial^2 p_i}{\partial y^2}\left(\frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\right) - \frac{g}{\rho_o}\left(\frac{\partial^2 \rho_i}{\partial y^2}\right) = 0 \quad . \tag{B.6}$$

Similarly,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial}{\partial z}\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}(\underline{f}\times\underline{u}_i)_z - \frac{\partial}{\partial z}\frac{\partial}{\partial x}(\underline{f}\times\underline{u}_i)_x$$

$$+ \frac{1}{\rho_o}\frac{\partial^2 \rho_i}{\partial x^2}\left(\frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\right) - \frac{g}{\rho_o}\left(\frac{\partial^2 \rho_i}{\partial x^2}\right) = 0 \quad . \tag{B.7}$$

Now

$$\frac{1}{\rho_o}\frac{\partial p_i}{\partial x} = -\frac{\partial u}{\partial t} - (\underline{f}\times\underline{u}_i)_x \tag{B.8}$$

and

$$\frac{1}{\rho_o}\frac{\partial p_i}{\partial y} = -\frac{\partial v}{\partial t} - (\underline{f}\times\underline{u}_i)_y \quad . \tag{B.9}$$

Using the incompressibility condition (B.5),

$$\frac{\partial}{\partial t}(\nabla^2 w) + \nabla^2(\underline{f}\times\underline{u}_i)_z - \frac{\partial}{\partial z}[\nabla\cdot(\underline{f}\times\underline{u}_i)]$$

$$-\frac{g}{\rho_o}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\rho_i + \frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\left[\frac{\partial}{\partial t}\frac{\partial w}{\partial z} - \nabla\cdot(\underline{f}\times\underline{u}_i)\right.$$

$$\left. + \frac{\partial}{\partial z}(\underline{f}\times\underline{u}_i)_z\right] = 0 \quad . \tag{B.10}$$

We now define the vorticity $\underline{\ell}$,

$$\underline{\ell} = \nabla\times\underline{u}_i \quad . \tag{B.11}$$

Thus

$$\nabla \cdot (\underline{f}\times\underline{u}_i) = -\underline{f}\cdot\underline{\ell} \tag{B.12}$$

and

$$\nabla^2(\underline{f}\times\underline{u}_i)_z = -\frac{\partial}{\partial z}(\underline{f}\cdot\underline{\ell}) + \nabla\cdot\left[\hat{z}\times\left(\nabla\times(\underline{f}\times\underline{u}_i)\right)\right] \quad . \tag{B.13}$$

This last term involves

$$\hat{z}\times\left[\nabla\times(\underline{f}\times\underline{u}_i)\right] = \hat{x}(\underline{f}\cdot\nabla)v - \hat{y}(\underline{f}\cdot\nabla)u \quad . \tag{B.14}$$

Thus

$$\frac{\partial}{\partial t}(\nabla^2 w) + \left(f_x\frac{\partial}{\partial y} - f_y\frac{\partial}{\partial x}\right)\frac{\partial w}{\partial z} + \left[f_x\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v\right.$$

$$\left. - f_y\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u\right] + f_z\frac{\partial \ell_z}{\partial z} - \frac{g}{\rho_o}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\rho_i$$

$$+ \frac{1}{\rho_o}\frac{\partial\rho_o}{\partial z}\left[\frac{\partial}{\partial t}\frac{\partial w}{\partial z} + \underline{f}\cdot\underline{\ell} + \frac{\partial}{\partial z}(\underline{f}\times\underline{u}_i)_z\right] = 0 \quad . \tag{B.15}$$

For $N^2 \gg g^2/c^2$,

$$N^2(z) \simeq \frac{g}{\rho_o}\frac{\partial\rho_o}{\partial z} \quad . \tag{B.16}$$

Thus, differentiating (B.15) with respect to time and employing the incompressibility condition (B.5),

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + N^2(z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w + \underline{f}\cdot\nabla\,\frac{\partial \ell_z}{\partial t}$$

$$= -\frac{N^2(z)}{g}\left[\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial z} + \underline{f}\cdot\frac{\partial \underline{\ell}}{\partial t} + \frac{\partial^2}{\partial t\partial z}(\underline{f}\times\underline{u}_i)_z\right] \quad . \tag{B.17}$$

The vorticity equation, which is the curl of equations (B.1), (B.2), and (B.3) added together, is

$$\frac{\partial \underline{\ell}}{\partial t} = -\nabla\times(\underline{f}\times\underline{u}_i) + \nabla\times\left(\frac{\rho_i g}{\rho_o}\hat{z}\right) \tag{B.18}$$

and, since

$$\nabla\times(\underline{f}\times\underline{u}_i) = -(\underline{f}\cdot\nabla)\underline{u}_i \quad , \tag{B.19}$$

we have

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + N^2(z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w + (\underline{f}\cdot\nabla)^2 w$$

$$= -\frac{N^2(z)}{g}\left[\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial z} + \underline{f}\cdot\left(\nabla\times\frac{\rho_i g}{\rho_o}\hat{z}\right)\right.$$

$$\left. + \underline{f}\cdot(\underline{f}\cdot\nabla)\underline{u}_i + \frac{\partial^2}{\partial t\partial z}(\underline{f}\times\underline{u}_i)_z\right] \quad . \tag{B.20}$$

APPENDIX C

Much of the following discussion of Markov processes comes from Ludwig Arnold's[9] marvelously lucid book and from lecture notes given by Werner Horsthemke.[57] More mathematical detail than is given in the discussion here may be found in these references.

The Markov property of stochastic dynamic systems says that "if the state of a system at a particular time s (the present) is known, additional information regarding the behavior of the system at times $t<s$ (the past) has no effect on our knowledge of the probable development of the system at $t>s$ (the future)".[58]

As an illustration, let us consider a discrete system, a machine which at the touch of a button chooses at random an integer from the closed interval [0,9]. This machine's inner workings will act so that the probability of getting any particular integer at the next push of the button depends only on what the last number was, and not on any of the numbers given before that. For example, say I have pushed the button three times at times t_1, t_2, and t_3 with the resulting sequence (9,4,6). Now, knowing that the last value was 6, the probability of getting a 7 is, say, 30%. That is,

$$P(7,t_4|6,t_3) = 0.3 \quad . \qquad (C.1)$$

Let us also say that (C.1) would have also been true if my previous trials had given (1,2,6), (8,5,6) or any sequence ending in 6. Then, our machine has the Markov property.

As an example of a non-Markovian system, let us say that in order to get a 7 after a 6, it is necessary to get a 5 before the 6 and, having gotten this 5-6 combination, I still have a 30% chance of getting a 7. Then

$$P\Big(7,t_4|(6,t_3),(5,t_2)\Big) = 0.3 \tag{C.2}$$

and, for example,

$$P\Big(7,t_4|(6,t_3),(8,t_2)\Big) = 0.0 \quad . \tag{C.3}$$

It is pretty hard to get a 7 after a 6 in this system! But the lack of the Markov property is clear; knowledge of the past improves our *prediction for the future.* (*Note: We do not consider the so-called* "higher-order Markovian systems" to be true Markov processes.)

Notice that it is not necessary for each trial to be totally (statistically) independent of all of the others for the system to be Markov. The Markov property simply states that given a set of trials and outcomes, $\Big((x_n,t_n),(x_{n-1},t_{n-1}),\ldots,(x_1,t_1)\Big)$,

$$P\Big(x_{n+1},t_{n+1}|(x_n,t_n),(x_{n-1},t_{n-1}),\ldots,(x_1,t_1)\Big) = P(x_{n+1},t_{n+1}|x_n,t_n) \quad , \tag{C.4}$$

$P(x_{n+1},t_{n+1}|x_n,t_n)$ is called the "transition probability".

The Markov property can also be true for continuous systems. Since we are talking about stochastic systems, we must be more specific about what is meant by "continuous", and that necessitates the concept of a "transition probability density". Consider a one-dimensional

system. If we know the transition probability $P(B,t|x,s)$ that the system will be in some interval B at time t if it is in state x at time $s<t$, the "transition probability density" $p(b,t|x,s)$(if it exists) is defined by

$$P(B,t|x,s) = \int_B db\ p(b,t|x,s) \quad . \tag{C.5}$$

We shall be concerned with systems that are continuous "almost surely":

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} dy\, p(y,t|x,s) = 0 \quad . \tag{C.6}$$

That is, as the length of time between measurements decreases to zero, the probability of finding the system in states separated by some difference ε goes to zero more quickly than $(t-s)$, no matter how small ε is. We are finally in a position to define a Markov diffusion process. A Markov diffusion process is an almost surely continuous Markov process where the "drift" $f(x,s)$ and "diffusion" $g^2(x,s)$, given by

$$f(x,s) = \lim_{t\downarrow s} \frac{1}{t-s} \int_{|y-x|\leq\varepsilon} dy\,(y-x)\ p(y,t|x,s) \quad , \tag{C.7}$$

$$g^2(x,s) = \lim_{t\downarrow s} \frac{1}{t-s} \int_{|y-x|\leq\varepsilon} dy\,(y-x)^2\ p(y,t|x,s) \quad , \tag{C.8}$$

exist. In many cases of interest, such as the cases we consider, the cutoff ε can be extended to the entire range of y.

The importance of these processes lies in that, since our system is a Markov diffusion process, we can immediately write a (backward) propagation equation for the moments. Often, as in the cases we shall consider, we can immediately write a (forward) propagation equation for the transition probability density, or "Fokker-Planck" equation:

$$\frac{dp(y,t|x,s)}{dt} = -\frac{\partial}{\partial y} f(y,t)\, p(y,t|x,s) + \frac{1}{2}\frac{\partial^2}{\partial y^2} g^2(y,t)\, p(y,t|x,s) \quad . \tag{C.9}$$

Now the reason our system is called a "*diffusion process*" is clear from the form of the Fokker-Planck equation, which is that characteristic of deterministic diffusion equations.

More general treatment of these concepts can be found in excellent books by Stratonovich,[53] Arnold,[9] or Horsthemke and Lefever.[59] In particular, we shall need the Fokker-Planck equation analogous to Eq. (C.9) for a system involving several coupled processes (the acoustic normal modes). Kohler and Papanicolaou[8] have shown the usefulness of the theory in acoustics and we shall apply it to the special case of acoustic propagation through a field of internal wave-induced sound speed fluctuations.

APPENDIX D

In terms of the components of $\underline{A}$, Eq. (5.28) becomes

$$\frac{dp(\underline{A},\underline{A}^*;\sigma|\underline{A}_o,\underline{A}_o^*;\sigma_o)}{d\sigma} = \lim_{T\to\infty} \frac{1}{T}\sum_{\substack{nm\\ n'm'}} \int_{\sigma_o}^{T+\sigma_o} ds \int_{\sigma_o}^{s} dt \Big\{ <V_{nm}(t)V_{n'm'}(s)>$$

$$\times \left[\frac{\partial}{\partial A_{m'}} A_n \delta_{n'm} - \frac{\partial}{\partial A_m}\frac{\partial}{\partial A_{m'}} A_n A_{n'}\right]$$

$$+ <V_{nm}^*(t)V_{n'm'}^*(s)> \left[\frac{\partial}{\partial A_{m'}^*} A_n^* \delta_{n'm} - \frac{\partial}{\partial A_m^*}\frac{\partial}{\partial A_{m'}^*} A_n^* A_{n'}^*\right]$$

$$+ <V_{nm}(t)V_{n'm'}^*(s)> \frac{\partial}{\partial A_m}\frac{\partial}{\partial A_{m'}^*} A_n A_{n'}^*$$

$$+ <V_{nm}^*(t)V_{n'm'}(s)> \frac{\partial}{\partial A_m^*}\frac{\partial}{\partial A_{m'}} A_n^* A_{n'} \Big\} p(\underline{A},\underline{A}^*;\sigma|\underline{A}_o,\underline{A}_o^*;\sigma_o) \quad , \qquad (D.1)$$

with $p(\underline{A},\underline{A}^*;\sigma_o|\underline{A}_o,\underline{A}_o^*;\sigma_o) = \delta(\underline{A}-\underline{A}_o)\delta(\underline{A}^*-\underline{A}_o^*)$.

To get a propagation equation for $<A_k>$, we multiply (D.1) by A_k and integrate over $\underline{A}$ and $\underline{A}^*$. Integration by parts gives us

$$\frac{d<A_k>}{d\sigma} = -\sum_{\substack{mn\\ m'n'}} \lim_{T\to\infty} \frac{1}{T}\int_{\sigma_o}^{\sigma_o+T} ds \int_{\sigma_o}^{s} dt <V_{nm}(t)V_{n'm'}(s)> <A_n> \delta_{n'm}\delta_{km'} \quad . \qquad (D.2)$$

From Eq. (5.14)

$$<V_{nm}(t)V_{n'm'}(s)> = \frac{\omega^4}{\varepsilon^2\sqrt{k_n k_m k_{n'} k_{m'}}}\int_0^H dz\int_0^H dz'\phi_n(z)\phi_m(z)\phi_{n'}(z')\phi_{m'}(z')$$

$$\times \rho(z)\rho(z')\,\frac{<\delta c(t,z)\delta c(s,z')>}{\bar{c}^3(z)\bar{c}^3(z')}\,e^{i(k_m-k_n)t}\,e^{i(k_{m'}-k_{n'})s}\quad . \qquad (D.3)$$

If s and t are in the same direction, i.e., the line connecting source and receiver,

$$\frac{<\delta c(t,z)\delta c(s,z')>}{\bar{c}(z)\bar{c}(z')} = \frac{8}{\pi^3}\left(\frac{\mu}{g}\right)^2 <\zeta^2(0)>\; N_0 f\,\frac{N^2(z)N^2(z')}{N(\eta)}$$

$$\times\int_0^{\pi/2} d\theta\int_f^{N(\eta)} d\Omega\int_{-\infty}^{\infty} d\alpha\,\frac{\alpha_*}{\alpha^2+\alpha_*^2}\,\frac{\left(\Omega^2-f^2\right)^{1/2}}{\Omega^3}\cos(\alpha\,\mathcal{z}\,X(\Omega))\,e^{i\alpha\cos\theta(t-s)}\quad , \qquad (D.4)$$

where $\eta = \frac{1}{2}(z+z')$, $\mathcal{z} = |z-z'|$, $X = \left(\frac{N^2(\eta)-\Omega^2}{\Omega^2-f^2}\right)^{1/2}$ and the other quantities are defined in Chapter 3.

Let $\Delta k_1 = k_m - k_n$, $\Delta k_2 = k_{m'} - k_{n'}$. Consider

$$I = \int_{\sigma_0}^{\sigma_0+T} ds\int_{\sigma_0}^{s} dt\; e^{i(\alpha\cos\theta+\Delta k_1)t}\, e^{i(\Delta k_2-\alpha\cos\theta)s}$$

$$
= - e^{i(\Delta k_1+\Delta k_2)\sigma_o} \frac{\left(e^{i(\Delta k_1+\Delta k_2)T} - 1\right)}{(\Delta k_1+\Delta k_2)} \frac{1}{(\alpha\cos\theta+\Delta k_1)}
$$

$$
+ e^{i(\Delta k_1+\Delta k_2)\sigma_o} \frac{\left(e^{i(\Delta k_2-\alpha\cos\theta)T} - 1\right)}{(\alpha\cos\theta+\Delta k_1)(\Delta k_2-\alpha\cos\theta)} \quad . \tag{D.5}
$$

Now we consider the α integration. Because of the operation $\lim_{T\uparrow\infty} 1/T$, only terms which vary at least linearly with T will survive. Hence, we exhibit only results of the α integration which meet that criterion.

From Eq. (D.5), I consists of two terms. Consider the first α integration,

$$
I_1 = - \frac{\left(e^{i(\Delta k_1+\Delta k_2)T} - 1\right)}{(\Delta k_1+\Delta k_2)} e^{i(\Delta k_1+\Delta k_2)\sigma_o} \int_{-\infty}^{\infty} d\alpha \frac{\alpha_\star}{\left(\alpha^2+\alpha_\star^2\right)} \frac{\cos(\alpha \not{z} X)}{(\alpha\cos\theta+\Delta k_1)} \quad . \tag{D.6}
$$

The only way that I can be proportional to T is if $\Delta k_1=-\Delta k_2$. Thus

$$
I_1 = - iT \int_{-\infty}^{\infty} d\alpha \frac{\alpha_\star}{(\alpha^2+\alpha_\star^2)} \frac{\cos[\alpha \not{z} X(\Omega)]}{\left(\alpha + \frac{\Delta k_1}{\cos\theta}\right)\cos\theta} \delta(\Delta k_1+\Delta k_2) \quad . \tag{D.7}
$$

Note that the pole at $\alpha = - \frac{\Delta k_1}{\cos\theta}$ appears to lie on the real axis. However, the quantity Δk_1 will have a small imaginary part if the ocean is slightly absorptive. We will use this imaginary part to lift the pole off the real axis and then, after evaluating the integral, let this imaginary part go to zero. Since $\Delta k_1=k_m-k_n$, Δk_1 will lie in the

fourth quadrant of the complex plane for n>m and in the second quadrant of the complex plane for n<m.[44] Thus,

$$I_1 = -\delta(\Delta k_1+\Delta k_2)\pi T \left\{ \frac{i\Delta k_1 \; e^{-\alpha_\star \mathscr{J} X} + \alpha_\star \; e^{i\Delta k_1 X \mathscr{J}/\cos\theta} \cos\theta}{(\alpha_\star^2 \cos^2\theta+\Delta k_1^2)} \right\}, \quad n<m \; , \qquad (D.8)$$

$$I_1 = -\delta(\Delta k_1+\Delta k_2)\pi T \left\{ \frac{i\Delta k_1 \; e^{-\alpha_\star X \mathscr{J}} - \alpha_\star \; e^{-i\Delta k_1 X \mathscr{J}/\cos\theta} \cos\theta}{(\alpha_\star^2 \cos^2\theta+\Delta k_1^2)} \right\}, \quad n>m \; . \qquad (D.9)$$

The second α integration is

$$I_2 = -e^{i(\Delta k_1+\Delta k_2)\sigma_0} \int_{-\infty}^{\infty} \frac{d\alpha \; \alpha_\star}{\cos^2\theta} \frac{\cos(\alpha \mathscr{J} X)\left(e^{i(\Delta k_2-\alpha\cos\theta)T} - 1\right)}{\left(\alpha^2+\alpha_\star^2\right)\left(\alpha + \frac{\Delta k_1}{\cos\theta}\right)\left(\alpha - \frac{\Delta k_2}{\cos\theta}\right)} \; . \qquad (D.10)$$

The only way (D.10) can be proportional to T is for the integral to contain a double pole; i.e., $\Delta k_1=-\Delta k_2$.

Now we see a problem. In Eq. (D.10) the quantity $X(\Omega)$ varies from zero to infinity <u>before</u> we take our limit $T\to\infty$. We would like to evaluate I_2 by contour integration and the relative sizes of $T\cos\theta$ and $\mathscr{J}X$ will make a difference in the way we close our contours. Thus, we must consider Eq. (D.4) in order to estimate the Ω integration between $\frac{T}{\mathscr{J}}\cos\theta$ and $X\to\infty$.

$$\int_f^N d\Omega \, \frac{\alpha_\star(\Omega)}{\left(\alpha^2+\alpha_\star^2\right)} \, \frac{\left(\Omega^2-f^2\right)^{1/2}}{\Omega^3} \cos(\alpha \, z X(\Omega))$$

$$= \int_0^\infty dX \, \frac{\alpha_\star(X)}{\left(\alpha^2+\alpha_\star^2\right)} \, \frac{\left(\Omega^2(X)-f^2\right)^{1/2}}{\Omega^3(X)} \cos(\alpha \, z X) \, \frac{\partial\Omega}{\partial X} \quad . \tag{D.11}$$

Define

$$\text{Rem} \equiv \left| \int_{\frac{T\cos\theta}{z}}^\infty dX \, \frac{\alpha_\star(X)}{\left(\alpha^2+\alpha_\star^2(X)\right)} \, \frac{\left(\Omega^2(X)-f^2\right)^{1/2}}{\Omega^3(X)} \cos(\alpha \, z X) \, \frac{\partial\Omega}{\partial X} \right| \quad , \tag{D.12}$$

$$\text{Rem} \leq \left| \int_{\frac{T\cos\theta}{z}}^\infty dX \, \frac{t(N^2-f^2)^2 \, X}{(N^2+X^2f^2)^2 \, (N^2t^2+\alpha^2X^2)} \right| \quad . \tag{D.13}$$

For f<<N but not zero, we have for nonzero α

$$\text{Rem} \lesssim \frac{tN^4}{4\alpha^2 f^4} \left(\frac{z}{T\cos\theta}\right)^4 \quad , \tag{D.14}$$

and for α=0

$$\text{Rem} \lesssim \frac{N^2}{2t^2f^2} \left|\frac{z}{T\cos\theta}\right|^2 \quad . \tag{D.15}$$

Since z can be no larger than the depth of the ocean, the remainder Rem is definitely small except in the region of $\theta=\pi/2$.

Let us define a small quantity δ such that $T\cos\theta < \mathcal{Z}X$ for $0<\theta<\pi/2-\delta$ and $\mathcal{Z}X > T\cos\theta$ for $\pi/2-\delta<\theta<\pi/2$.

Then, dropping the terms independent of T,

$$\int_0^{\pi/2} d\theta\ I_2 = -\delta(\Delta k_1+\Delta k_2)\, e^{i\Delta k_2 T} \left\{ \int_0^{\pi/2-\delta} d\theta \int_{-\infty}^{\infty} d\alpha\, \frac{\alpha_\star\, e^{-i\alpha\cos\theta T} \cos(\alpha \mathcal{Z} X)}{\cos^2\theta(\alpha^2+\alpha_\star^2)\left(\alpha + \frac{\Delta k_1}{\cos\theta}\right)^2} \right.$$

$$\left. + \frac{1}{2}\int_{\pi/2-\delta}^{\pi/2} d\theta \int_{-\infty}^{\infty} d\alpha\, \frac{\alpha_\star\left(e^{i\alpha(\mathcal{Z}X-T\cos\theta)} + e^{-i\alpha(\mathcal{Z}X+T\cos\theta)}\right)}{\left(\alpha^2+\alpha_\star^2\right)\left(\alpha + \frac{\Delta k_1}{\cos\theta}\right)^2 \cos^2\theta} \right\}, \qquad (D.16)$$

and

$$\frac{1}{T}\int_0^{\pi/2} d\theta I_2 = 2\pi\delta(\Delta k_1+\Delta k_2)\left\{ \int_0^{\pi/2-\delta} d\theta\, \frac{\cos\theta\cos\left(\frac{\Delta k_1 \mathcal{Z} X}{\cos\theta}\right)}{\left(\alpha_\star^2\cos^2\theta+\Delta k_1^2\right)} \right.$$

$$\left. + O\left(\frac{1}{T}\right) + O\left(\frac{\mathcal{Z}X\delta}{T}\right) \right\} \qquad . \qquad (D.17)$$

We may therefore neglect the remainder term for very small δ. Physically, we expect this. The integration procedure combined with the limit on T picks out of the internal wave spectrum those radial wave number components (components in the direction of acoustic propagation) equal to $|k_m-k_n|$. By neglecting the term $O\left(\frac{\mathcal{Z}X\delta}{T}\right)$ we say that internal waves traveling in a direction orthogonal to the direction of acoustic propagation do not cause scattering of energy among the

acoustic depth modes. That is, the probability of finding internal waves with infinite wave number is zero.

If we impose the condition that acoustic eigenvalues k along with their sums and differences exist uniquely, then

$$\delta(\Delta k_1+\Delta k_2) = \delta_{nm'}\delta_{mn'} \tag{D.18}$$

and

$$\frac{d\langle A_k\rangle}{d\sigma} = -\frac{1}{\varepsilon^2}\sum_m (a_{mk}-ib_{mk})\langle A_k\rangle \quad , \tag{D.19}$$

where

$$a_{mk} = \frac{8}{\pi^2}\left(\frac{\mu}{g}\right)^2 \langle\zeta^2(0)\rangle \; N_o f \; \frac{\omega^4}{k_m k_k}\int_0^H dz \int_0^H dz'\rho(z)\rho(z')\phi_m(z)\phi_k(z)$$

$$\times \phi_m(z')\phi_k(z') \; \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(n)}\int_f^{N(n)} d\Omega \; \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3} \; \alpha_*$$

$$\times \int_0^{\pi/2} d\theta \; \frac{\cos\theta\cos\left[\frac{(k_m-k_k)\,\mathcal{J}X(\Omega)}{\cos\theta}\right]}{\left[\alpha_*^2\cos^2\theta+(k_m-k_k)^2\right]} \quad , \tag{D.20}$$

and

$$b_{mk} = \frac{8}{\pi^2}\left(\frac{\mu}{g}\right)^2 \langle\zeta^2(0)\rangle \quad N_o f \; \frac{\omega^4}{k_m k_k}\int_0^H dz\int_0^H dz'\rho(z)\rho(z')\phi_m(z)\phi_k(z)\phi_m(z')\phi_k(z')$$

$$\times \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(n)} \int_f^{N(n)} d\Omega \; \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3}$$

$$\times \int_0^{\pi/2} d\theta \; \frac{\left\{(k_m-k_k)\; e^{-\alpha_* \mathcal{J} X} + \alpha_* \cos\theta \sin\left[\frac{(k_m-k_k)\mathcal{J} X}{\cos\theta}\right]\right\}}{\left[\alpha_*^2 \cos\theta + (k_m-k_k)^2\right]} \quad . \qquad (D.21)$$

Equation (D.19) has the solution

$$< A_k(\frac{\sigma}{\varepsilon^2}) > \; = \exp\left[-\Lambda_k \frac{\sigma - \sigma_o}{\varepsilon^2}\right] < A_k(\frac{\sigma_o}{\varepsilon^2}) > \quad , \qquad (D.22)$$

where

$$\Lambda_k = \sum_m (a_{mk} - ib_{mk}) \quad . \qquad (D.23)$$

Note that even with the oscillatory terms $a_{mk}>0$ and is symmetric in m and k while b_{mk} is antisymmetric in m and k. Note also that Λ_k is independent of σ_o.

APPENDIX E

Consider

$$\frac{2\sin\left[\frac{\pi}{2}(\Delta q+\Delta q')\right]}{(\Delta q+\Delta q')}\int_{-\infty}^{\infty} d\alpha \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' P(\alpha,\theta;\eta)\cos(\alpha z X)$$

$$\times e^{i(\Delta q\varphi+\Delta q'\varphi')} e^{i\alpha\sigma\cos\varphi} e^{-i\alpha t\cos\varphi'} \quad . \qquad \text{(E.1)}$$

Since Δq, $\Delta q'$ are integers, the expression (E.1) is equal to

$$\frac{8\sin\left[\frac{\pi}{2}(\Delta q+\Delta q')\right]}{(\Delta q+\Delta q')}\int_{-\infty}^{\infty} d\alpha \int_0^{\pi} d\varphi \int_0^{\pi} d\varphi' P(\alpha,\theta;\eta)\cos(\alpha z X)$$

$$\times e^{i\alpha\sigma\cos\varphi}\cos(\Delta q\varphi)\, e^{-i\alpha t\cos\varphi'}\cos(\Delta q'\varphi') \quad . \qquad \text{(E.2)}$$

Now

$$\int_0^{\pi} d\varphi\, e^{i\alpha\sigma\cos\varphi}\cos(\Delta q\varphi) = \int_0^{\pi/2} d\varphi\, e^{i\alpha\sigma\cos\varphi}\cos(\Delta q\varphi)$$

$$+\int_{\pi/2}^{\pi} d\varphi\, e^{i\alpha\sigma\cos\varphi}\cos(\Delta q\varphi) \quad . \qquad \text{(E.3)}$$

In the second integral, let $\varphi \to -\varphi$, so

$$\int_0^{\pi} d\varphi\, e^{i\alpha\sigma\cos\varphi} \cos(\Delta q\varphi) = \int_0^{\pi/2} d\varphi\, e^{i\alpha\sigma\cos\varphi} \cos(\Delta q\varphi)$$

$$+ \cos(\Delta q\pi) \int_0^{\pi/2} d\varphi\, e^{-i\alpha\sigma\cos\varphi} \cos(\Delta q\varphi) \quad . \tag{E.4}$$

If we perform similar manipulations with the φ' integral,

$$\frac{8 \sin\left[\frac{\pi}{2}(\Delta q+\Delta q')\right]}{(\Delta q+\Delta q')} \int_{-\infty}^{\infty} d\alpha \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\varphi'\, P(\alpha,\Omega;n) \cos(\alpha \mathfrak{z} X)$$

$$\times \left[e^{i\alpha\sigma\cos\varphi} \cos(\Delta q\varphi) + \cos(\Delta q\pi)\, e^{-i\alpha\sigma\cos\varphi} \cos(\Delta q\varphi)\right]$$

$$\times \left[e^{-i\alpha t\cos\varphi'} \cos(\Delta q'\varphi') + \cos(\Delta q'\pi)\, e^{i\alpha t\cos\varphi'} \cos(\Delta q'\varphi')\right] \quad . \tag{E.5}$$

Since $P(\alpha,\Omega;n) \cos(\alpha \mathfrak{z} X)$ is even in α, the integration over α will have nonzero contribution only if Δq and $\Delta q'$ are both even or both odd. However, if they are not also equal in magnitude and opposite in sign, the sine term will vanish. Thus,

$$\Delta q = -\Delta q' \tag{E.6}$$

and

$$\frac{2\sin\left[\frac{\pi}{2}(\Delta q+\Delta q')\right]}{(\Delta q+\Delta q')}\int_{-\infty}^{\infty}d\alpha\int_{0}^{2\pi}d\varphi\int_{0}^{2\pi}d\varphi'\,P(\alpha,\Omega;n)\,\cos(\alpha \mathcal{Z} X)$$

$$\times\, e^{i(\Delta q\varphi+\Delta q'\varphi')}\, e^{i\alpha\sigma\cos\varphi}\, e^{-i\alpha t\cos\varphi'} =$$

$$\pi\delta(\Delta q+\Delta q')\int_{-\infty}^{\infty}d\alpha\int_{0}^{2\pi}d\varphi\int_{0}^{2\pi}d\varphi'\,P(\alpha,\Omega;n)\,\cos(\alpha \mathcal{Z} X)$$

$$\times\, e^{i\Delta q(\varphi-\varphi')}\, e^{i\alpha\sigma\cos\varphi}\, e^{-i\alpha t\cos\varphi'} \quad . \tag{E.7}$$

APPENDIX F

Consider the expression

$$\lim_{t\uparrow\infty} \frac{1}{T}\int_{\sigma_0}^{\sigma_0+T} ds\int_{\sigma_0}^{s} dt\int_{-\infty}^{\infty} d\alpha P(\alpha,\Omega;\eta)\ \cos(\alpha\mathscr{z}X)\, e^{i\alpha(s\cos\varphi-t\cos\varphi')}$$

$$\times\, e^{i\Delta k_1 s}\, e^{i\Delta k_2 t}\, \exp\left(i\frac{\Delta Q_1^2}{2s}\right)\exp\left(i\frac{\Delta Q_2^2}{2t}\right) \tag{F.1}$$

$$=\sum_{j=0}^{\infty} \lim_{T\uparrow\infty} \frac{1}{T}\int_{\sigma_0}^{\sigma_0+T} ds\int_{\sigma_0}^{s} dt\int_{-\infty}^{\infty} d\alpha P(\alpha,\Omega;\eta)\ \cos(\alpha\mathscr{z}X)$$

$$\times\, e^{i\alpha s\cos\varphi}\, e^{-i\alpha t\cos\varphi'}\, e^{i(\Delta k_1 s+\Delta k_2 t)}\, \frac{1}{j!}\left(\frac{\Delta Q_1^2}{2s}+\frac{\Delta Q_2^2}{2t}\right)^j \quad . \tag{F.2}$$

The j=1 term is

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_{-\infty}^{\infty} d\alpha P(\alpha,\Omega;\eta)\ \cos(\alpha\mathscr{z}$$

$$\times \int_{\sigma_0}^{\sigma_0+T} ds\int_{\sigma_0}^{s} dt\ e^{is(\Delta k_1+\alpha\cos\varphi)}\, e^{it(\Delta k_2-\alpha\cos\varphi')}\left(\frac{\Delta Q_1^2}{2s}+\frac{\Delta Q_2^2}{2t}\right) \tag{F.3}$$

$$= \lim_{T\uparrow\infty} \frac{1}{2T} \int_{-\infty}^{\infty} d\alpha P(\alpha,\Omega;n)\, \cos(\alpha \mathfrak{z} X)$$

$$\times \left\{ \left(\frac{\Delta Q_1^2}{i(\Delta k_2 - \alpha\cos\varphi')} - \frac{\Delta Q_2^2}{i(\Delta k_1 + \alpha\cos\varphi)} \right) \left\{ \mathrm{Ei}\Big[i(\sigma_0 + T)\big(\Delta k_1 + \Delta k_2 + \alpha[\cos\varphi - \cos\varphi']\big) \Big] \right.\right.$$

$$\left.\left. - \mathrm{Ei}\Big[i\sigma_0 \big(\Delta k_1 + \Delta k_2 + \alpha[\cos\varphi - \cos\varphi']\big) \Big] \right\}\right.$$

$$- \frac{\Delta Q_1^2\, e^{i(\Delta k_2 - \alpha\cos\varphi')\sigma_0}}{2i(\Delta k_2 - \alpha\cos\varphi')} \left\{ \mathrm{Ei}[i(\sigma_0 + T)(\Delta k_1 + \alpha\cos\varphi)] - \mathrm{Ei}[i\sigma_0(\Delta k_1 + \alpha\cos\varphi)] \right\}$$

$$\left. + \frac{\Delta Q_2^2\, e^{i(\Delta k_1 + \alpha\cos\varphi)(\sigma_0 + T)}}{2i(\Delta k_1 + \alpha\cos\varphi)} \left\{ \mathrm{Ei}[i(\sigma_0 + T)(\Delta k_2 - \alpha\cos\varphi')] - \mathrm{Ei}[i\sigma_0(\Delta k_2 - \alpha\cos\varphi')] \right\} \right\} \tag{F.4}$$

$$\leqslant o\left(\frac{\ln T}{T}\right) + O\left(\frac{1}{T}\right) \quad , \tag{F.5}$$

since

$$\left| \frac{\alpha_\star}{\alpha^2 + \alpha_\star^2} \cos(\alpha \mathfrak{z} X) \right| \leqslant \frac{1}{\alpha_\star} \quad . \tag{F.6}$$

Thus, in taking the limit $T \uparrow \infty$, the $j=1$ term and, similarly, the $j \geq 2$ terms vanish as long as ΔQ_1^2 and ΔQ_2^2 are finite.

APPENDIX G

From Eq. (6.22) and the results of Appendix F,

$$\frac{d\langle A_{kj}A^*_{k'j'}\rangle}{d\sigma} = \lim_{T\to\infty}\frac{1}{T}\sum_{\substack{mpnq\\ m'p'n'q'}}\Big\{-\mathcal{L}_1\langle A_{nq}A^*_{k'j'}\rangle\,\delta_{p',q-p+q'}\delta_{m'k}\delta_{p'j}\delta_{pq'}$$

$$-\mathcal{L}_2\langle A_{kj}A^*_{nq}\rangle\,\delta_{mn'}\delta_{m'k'}\delta_{p'j'}\delta_{p',q-p+q'}$$

$$+\mathcal{L}_3\langle A_{nq}A^*_{n'q'}\rangle\,\delta_{mk}\delta_{pj}\delta_{m'k'}\delta_{p'j'}\delta_{p',q'-q+p}$$

$$+\mathcal{L}_4\langle A_{n'q'}A^*_{nq}\rangle\,\delta_{mk'}\delta_{pj'}\delta_{m'k}\delta_{p'j}\delta_{p',q'-q+p}\Big\}\quad. \tag{G.1}$$

We will consider each of the $\mathcal{L}_i$'s in turn.

$$\mathcal{L}_1 = \frac{\omega^4}{\varepsilon^2\pi^4}\left(\frac{\mu}{g}\right)^2\frac{N_0 f\langle\zeta^2(0)\rangle}{\sqrt{k_m k_n k_{m'} k_{n'}}}\int_0^{2\pi}d\varphi\int_0^{2\pi}d\varphi'\int_0^{H}dz\int_0^{H}dz'\rho(z)\rho(z')$$

$$\times\ \phi_n(z)\phi_m(z)\phi_{n'}(z')\phi_{m'}(z')\ \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(n)}$$

$$\times\int_f^{N(n)}d\Omega\int_{-\infty}^{\infty}d\alpha\ \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3}\ \alpha_*\ \frac{\cos(\alpha\, \not{g}X)}{(\alpha^2+\alpha_*^2)}$$

$$\times \int_{\sigma_o}^{\sigma_o+T} ds \int_{\sigma_o}^{s} dt \ e^{i(p-q)\varphi} \ e^{i(p'-q')\varphi'}$$

$$\times \ e^{i(k_m - k_n)s} \ e^{i(k_{m'} - k_{n'})t} \ e^{i\alpha(s\cos\varphi - t\cos\varphi')} \quad . \qquad (G.2)$$

Again we see the operator evaluated in Appendix D and

$$\lim_{T\uparrow\infty} \frac{1}{T} \sum_{\substack{mpnq \\ m'p'n'q'}} \mathcal{L}_1 \langle A_{nq} A^{*}_{k'j'} \rangle \ \delta_{p',q-p+q'} \delta_{m'k} \delta_{p'j} \delta_{mn'} \delta_{pq'} =$$

$$- \frac{1}{\varepsilon^2} \sum_m (a_{mk} - ib_{mk}) \langle A_{kj} A^{*}_{k'j'} \rangle \quad . \qquad (G.3)$$

The second term is the complex conjugate of the first with an exchange of indices $(k,j) \leftrightarrow (k',j')$:

$$\lim_{T\uparrow\infty} \frac{1}{T} \sum_{\substack{mpnq \\ m'p'n'q'}} \mathcal{L}_2 \langle A^{*}_{nq} A_{kj} \rangle \delta_{mn'} \delta_{pq'} \delta_{m'k'} \delta_{p'j'} \delta_{p',q-p+q'} =$$

$$- \frac{1}{\varepsilon^2} \sum_m (a_{mk'} + ib_{mk'}) \langle A_{kj} A^{*}_{k'j'} \rangle \quad . \qquad (G.4)$$

Calculation of the third term is a bit trickier:

$$\sum_{\substack{mpnq \\ m'p'n'q'}} \mathcal{L}_3 \ \langle A_{nq} A^{*}_{n'q'} \rangle \ \delta_{mk} \delta_{pj} \delta_{m'k'} \delta_{p'j'} \delta_{p',q'-q+p} =$$

$$\sum_{\substack{nq\\n'q'}} \frac{\omega^4}{\varepsilon^2\pi^4}\left(\frac{\mu}{g}\right)^2 N_o f <\zeta^2(0)> \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \int_0^H dz \int_0^H dz'\rho(z)\rho(z')$$

$$\times \frac{\phi_n(z)\phi_k(z)\phi_{n'}(z')\phi_{k'}(z')N^2(z)N^2(z')}{\sqrt{k_n k_k k_{n'} k_{k'}}\ \ \bar{c}^2(z)\bar{c}^2(z')N(n)} \int_f^{N(n)} d\Omega \int_{-\infty}^{\infty} d\alpha\ \cos(\alpha \mathcal{J} X)$$

$$\times \frac{(\Omega^2-f^2)^{1/2}}{\Omega^3} \frac{\alpha_*}{\alpha^2+\alpha_*^2} \int_{\sigma_o}^{\sigma_o+T} ds \int_{\sigma_o}^{s} dt\ e^{i(j-q)\varphi}\ e^{-i(j'-q')\varphi}$$

$$\times e^{i(k_k-k_n)s}\ e^{-i(k_{k'}-k_{n'})t}\ e^{i\alpha(s\cos\varphi - t\cos\varphi')} < A_{nq}A^*_{n'q'} > \delta_{j'-j,q'-q} \quad . \tag{G.5}$$

A similar term is obtained for the fourth term. It is here that excitation of the azimuthal modes is explicit. Even if only the azimuthally symmetric mode is initially nonzero, all of the other azimuthal modes are excited as long as j-j' is zero. This null difference between j and j' is maintained throughout the entire field of propagation and the expression for $<|p|^2>$, for example, becomes

$$<|p|^2> = \omega^2\rho^2 \sum_{k,} \frac{<A_{kj}A^*_{k'j}>}{r\sqrt{k_k k_{k'}}} \phi_k(z)\phi_{k'}(z')$$

$$\times \exp\left\{ i\left[(k_k-k_{k'})r + \frac{j^2}{2r}\left(\frac{1}{k_k} - \frac{1}{k_{k'}}\right)\right]\right\} \quad . \tag{G.6}$$

That is, for an azimuthally symmetric source, the different azimuthal modes are uncorrelated to first order in ε.

After much manipulation, we find that

$$\frac{d\langle A_{kj}A^*_{k'j}\rangle}{d\sigma} = -\frac{1}{\varepsilon^2}\sum_m \left[(a_{mk}+a_{mk'}) + i(b_{mk'}-b_{mk})\right]\langle A_{kj}A^*_{k'j}\rangle$$

$$+\frac{1}{\varepsilon^2}\sum_{\substack{mm'\\q}} d^{qj}_{mkm'k'}\left(\langle A_{mq}A^*_{m'q}\rangle + \langle A_{m'q}A^*_{mq}\rangle\right) \quad , \qquad (G.7)$$

where the sums over m and m' are restricted such that $|k_k-k_m| \leqslant |k_{k'}-k_{m'}|$ and that $(k_{k'}-k_{m'})$ has the same sign as (k_k-k_m). The quantities $d^{qj}_{mkm'k'}$ are given by

$$d^{qj}_{mkm'k'} = \frac{16\omega^4}{c^3}\left(\frac{\mu}{g}\right)^2 N_o f \langle\zeta^2(0)\rangle \int_0^H dz \int_0^H dz' \, \frac{\phi_m(z)\phi_{m'}(z')\phi_k(z)\phi_{k'}(z')}{\sqrt{k_m k_{m'} k_k k_{k'}}}$$

$$\times \frac{N^2(z)N^2(z')}{\bar{c}^2(z)\bar{c}^2(z')N(\eta)}\,\rho(z)\rho(z')\int_f^{N(\eta)} d\Omega\,\frac{(\Omega^2-f^2)^{1/2}}{\Omega^3}\,\alpha_*$$

$$\times \int_0^{\pi/2} d\varphi \cos\varphi \cos[(j-q)\varphi]\,\cos\left[(j-q)\cos^{-1}\left(\frac{k_k-k_m}{k_{k'}-k_{m'}}\cos\varphi\right)\right]$$

$$\times \cos\frac{\left(\frac{k_{k'}-k_{m'}}{\cos\varphi} \quad \mathcal{J}X\right)}{\left(\alpha_*^2\cos^2\varphi + (k_{k'}-k_{m'})^2\right)} \quad . \qquad (G.8)$$

The expression (G.7) is prohibitively difficult to calculate since the $d^{qj}_{mkm'k'}$ terms would have to be evaluated $M^4(2L+1)^2$ times in order to specify the propagation of each $<A_{kj}A^*_{k'j}>$ term! However, much information can be obtained about energy transfer without actually solving for each $<A_{kj}A^*_{k'j}>$. From Eq. (G.6), we see that $<|p|^2>$ (and also $<\underline{j}>$) contains terms

$$w_k = \sum_j <|A_{kj}|^2> \qquad . \tag{G.9}$$

If we sum Eq. (G.7) over j we find that

$$\frac{dw_k}{d\sigma} = \frac{2}{\varepsilon^2}\sum_m a_{mk}(w_m - w_k) \tag{G.10}$$

if k=k' and

$$\frac{d}{d\sigma}\sum_j <A_{kj}A^*_{k'j}> = -\frac{1}{\varepsilon^2}\sum_m \Big[(a_{mk}+a_{mk'}) + i(b_{mk'}-b_{mk})\Big]\sum_j <A_{kj}A^*_{k'j}>$$

for k≠k'.

APPENDIX H

Evaluation of the Frequency Integral

Using partial fractions,

$$\frac{1}{(X^2+K^2)^2 (X^2+Q^2)} = \frac{1}{(Q^2-K^2)(X^2+K^2)^2} + \frac{1}{(Q^2-K^2)^2}\left[\frac{1}{(X^2+Q^2)} - \frac{1}{(X^2+K^2)}\right] \quad . \tag{H.1}$$

So,

$$\int_0^\infty dX \, \frac{X\cos(GX)}{(X^2+K^2)^2 (X^2+Q^2)} = \frac{1}{(Q^2-K^2)} \int_0^\infty dX \, \frac{X\cos(GX)}{(X^2+K^2)^2}$$

$$+ \frac{1}{(Q^2-K^2)^2} \int_0^\infty dX \, \frac{X\cos(GX)}{(X^2+Q^2)} - \frac{1}{(Q^2-K^2)^2} \int_0^\infty dX \, \frac{X\cos(GX)}{(X^2+K^2)} \quad . \tag{H.2}$$

We integrate the first term by parts. Then, with the help of Gradshteyn and Ryzhik,[60,61]

$$\int_0^\infty dX \, \frac{X\cos(GX)}{(X^2+K^2)^2 (X^2+Q^2)} = \frac{1}{2(Q^2-K^2)K^2} - \frac{G}{4K(Q^2-K^2)}\left[e^{-GK}\,\mathrm{Ei}(GK) - e^{GK}\,\mathrm{Ei}(-GK)\right]$$

$$+ \frac{1}{2(Q^2-K^2)^2}\left[e^{-GK}\,\mathrm{Ei}(GK) + e^{GK}\,\mathrm{Ei}(-GK)\right]$$

$$- \frac{1}{2(Q^2-K^2)^2}\left[e^{-GQ}\,\mathrm{Ei}(GQ) + e^{GQ}\,\mathrm{Ei}(-GQ)\right] \quad . \tag{H.3}$$

For $G \to 0$, (H.3) becomes

$$\int_0^\infty dX \frac{X}{(X^2+K^2)^2(X^2+Q^2)} = \frac{1}{2(Q^2-K^2)K^2} + \frac{1}{(Q^2-K^2)^2}(\ln K - \ln Q) \quad . \qquad (H.4)$$

Contour Evaluation of the Integral when G=0

The integral of interest is

$$I_1 = \int_0^\infty dX \frac{X}{(X^2+K^2)^2(X^2+Q^2)} \quad . \qquad (H.5)$$

Consider instead the integral

$$I_2 = \oint dz \frac{z \ln z}{(z^2+K^2)^2(z^2+Q^2)} \quad , \qquad (H.6)$$

where we choose the contour as in Fig. 8. Thus,

$$I_2 = \int_0^\infty dr \frac{r \ln r}{(r^2+K^2)^2(r^2+Q^2)} + \int_\infty^0 dr \frac{r(\ln r + 2\pi i)}{(r^2+K^2)^2(r^2+Q^2)} \quad . \qquad (H.7)$$

The ln r terms cancel and, since

$$- 2\pi i I_1 = 2\pi i \sum \text{Residues } (I_2) \quad , \qquad (H.8)$$

the result (H.4) follows immediately upon evaluating the residues of I_2.

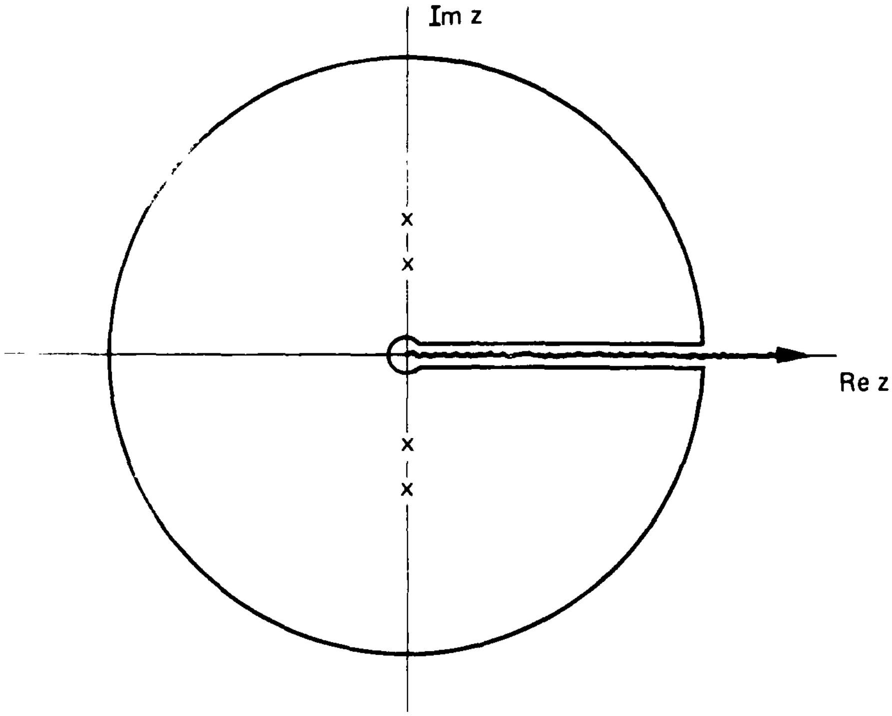


FIGURE 8
CONTOUR FOR EVALUATION OF I_2 SHOWING POLES ON THE IMAGINARY AXIS AND THE BRANCH LINE ALONG THE POSITIVE REAL AXIS.

ARL:UT
AS-84-825
CMD - GA
9 - 24 - 84

REFERENCES

1. C. Garrett and W. Munk, "Internal Waves in the Ocean," A. Rev. Fluid Mech. 11, 339-369 (1979).
2. C. Garrett and W. Munk, "Space-Time Scales of Internal Waves," Geophys. Fluid Dynam. 2, 225-264 (1962).
3. M. Briscoe, "Internal Waves in the Ocean," Rev. Geophys. Space Phys. 13, 591-598 (1975).
4. S. Flatté, R. Dashen, W. Munk, K. Watson, and F. Zachariason, Sound Transmission Through a Fluctuating Ocean (Cambridge University Press, Cambridge, London, 1979).
5. S. Flatté, "Wave Propagation Through Random Media: Contributions from Ocean Acoustics," Proc. IEEE 71, 1267-1294 (1983).
6. L. Dozier and F. Tappert, "Statistics of Normal Mode Amplitudes in a Random Ocean. I. Theory," J. Acoust. Soc. Am. 63, 353-365 (1978).
7. L. Dozier and F. Tappert, "Statistics of Normal Mode Amplitudes in a Random Ocean. II. Computations," J. Acoust. Soc. Am. 64, 533-547 (1978).
8. W. Kohler and G. Papanicolaou, in Lecture Notes in Physics, Vol. 70, Wave Propagation and Underwater Acoustics (Springer-Verlag, Berlin, 1977) edited by J. Keller and J. Papadakis, Chap. 4, pp. 153-223.
9. L. Arnold, Stochastic Differential Equations: Theory and Applications (John Wiley and Sons, New York, 1974).
10. G. Papanicolaou and W. Kohler, "Asymptotic Theory of Mixing Stochastic Ordinary Differential Equations," Commun. Pure Appl. Math. 27, 641-668 (1974).

11. C. Eckart, Hydrodynamics of Oceans and Atmospheres (Pergamon Press, New York, 1960).

12. C. Eckart, The Sea, Vol. 1, edited by M. N. Hill (Interscience Publishers, New York, 1962) Chap. 2, pp. 31-41.

13. H. Lamb, Hydrodynamics, 6th edition (Dover Publications, New York, 1945) Chap. 9, p. 574.

14. V. Väisälä, "Über die Wirkung der Windschwankungen auf die Pilot beobachtungen," Soc. Sci. Fennica, Commentationes Phys.-Math. 19, 37 (1925).

15. J. Lighthill, Waves in Fluids, 1st paperback edition (Cambridge University Press, Cambridge, London, 1979), Chap. 4, pp. 287-309.

16. C. Eckart, op. cit. Ref. 11, Chap. 4, pp. 59-63.

17. C. Eckart, op. cit. Ref. 11, Chap. 10, p. 148.

18. R. J. Urick, Principles of Underwater Sound, 2nd edition (McGraw-Hill Book Co., Inc., New York, 1975), Chap. 6, pp. 136-139.

19. W. Munk, "Sound Channel in an Exponentially Stratified Ocean with Application to SOFAR," J. Acoust. Soc. Am. 55, 220-226 (1974).

20. J. Reid and R. Lynn, "On the Influence of the Norwegian-Greenland and Weddell Seas upon the Bottom Waters of the Indian and North-Pacific Oceans," Deep Sea Res. 18, 1063-1088 (1971).

21. D. Pingree and G. Morrison, "The Relationship Between Stability and Source Waters for a Section of the Northeast Atlantic," J. Phys. Oceangr. 3, 280-285 (1973).

22. C. Eckart, op. cit. Ref. 11, Chap. 5, pp. 64-71.

23. C. Eckart, op. cit. Ref. 11, Chap. 7, pp. 94-97.

24. S. Flatté et al., op. cit. Ref. 4, pp. 44-49.

25. O. Phillips, The Dynamics of the Upper Ocean (Cambridge University Press, Cambridge, London, 1969) Chap. 5, pp. 161-165.

26. O. Phillips, ibid, pp. 190-197.

27. S. Flatté et al., op. cit. Ref. 4, pp. 54-57.

28. D. Olbers, "Models of the Oceanic Internal Wave Field," Rev. Geophys. Space Phys. 21, 1567-1606 (1983).

29. Y. Desaubies, private communication (1983).

30. K. Leaman and T. Sanford, "Vertical Energy Propagation of Internal Waves: A Vector Spectral Analysis of Velocity Profiles," J. Geophys. Res. 80, 1975-1978 (1975).

31. T. Sanford, "Observations of the Vertical Structure of Internal Waves," J. Geophys. Res. 80, 3861-3871 (1975).

32. P. Müller, D. Olbers, and J. Willebrand, "The IWEX Spectrum," J. Geophys. Res. 80, 479-500 (1975).

33. C. Garrett and W. Munk, "Space-Time Scales of Internal Waves: A Progress Report," J. Geophys. Res. 80, 291-297 (1975).

34. M. Briscoe, "Preliminary Results from the Trimoored Internal Wave Experiment (IWEX)," J. Geophys. Res. 80, 3872-3886 (1975).

35. J. Cairns and G. Williams, "Internal Wave Measurements from a Midwater Float, 2," J. Geophys. Res. 81, 1943-1950 (1976).

36. Y. Desaubies, "Analytical Representation of Internal Wave Spectra," J. Phys. Oceanogr. 6, 976-981 (1976).

37. S. Flatté et al., op. cit. Ref. 4, pp. 59-61.

38. A. Voorhis, "Measurements of Vertical Motion and the Partition of Energy in the New England Slope Water," Deep Sea Res. 15, 599-608 (1968).

39. W. Gould, "Spectral Characteristics of some Deep Current Records from the Eastern North Atlantic," Philos. Trans. R. Soc. London, Ser. A, 270, 437-450 (1971).

40. Y. Desaubies, "A Linear Theory of Internal Wave Spectra and Coherences near the Väisälä Frequency," J. Geophys. Res. 80, 895-899 (1975).

41. Y. Desaubies, "Internal Waves near the Turning Point," Geophys. Fluid Dyn. 5, 143-154 (1973).

42. I. Tolstoy and C. Clay, Ocean Acoustics (McGraw-Hill Book Co., Inc., New York, 1966).

43. E. Titchmarsh, Eigenfunction Expansions, 2nd edition (Oxford University Press, London, 1962).

44. R. Koch, C. Penland, P. Vidmar, and K. Hawker, "On the Calculation of Normal Mode Group Velocity and Attenuation," J. Acoust. Soc. Am. 73, 820-825 (1983).

45. R. Pitre, "On the Application of Horizontal Ray Theory to Acoustic Propagation in the Oceanic Waveguide," Ph.D. Thesis, The University of Texas at Austin, May, 1984.

46. B. Auld, Acoustic Fields and Waves in Solids (John Wiley and Sons, New York, 1973), pp. 154-158.

47. J. Watson, W. Siegmann, and M. Jacobson, "Acoustically Relevant Statistics for Stochastic Internal Wave Spectra," J. Acoust. Soc. Am. 61, 716-725 (1977).

48. S. Flatté et al., op. cit. Ref. 4, Chap. 5, pp. 76-78.

49. F. Tappert, in Lecture Notes in Physics, Vol. 70, Wave Propagation and Underwater Acoustics (Springer-Verlag, Berlin, 1977), edited by J. Keller and J. Papadakis, Chap. 5, pp. 224-287.

50. G. Blankenship and G. Papanicolaou, "Stability of Stochastic Systems with Wide-Band Noise Disturbances I," SIAM J. Appl. Math. 34, 437-476 (1978).

51. W. Kohler, "Pulse Propagation in a Randomly Perturbed Ocean: Single Pulse Statistics," J. Acoust. Soc. Am. 68, 1177-1183 (1980).

52. J. Morrison and J. McKenna, Bell Syst. Tech. J. 51, 209 (1972).

53. R. Stratonovich, Topics in the Theory of Random Noise (Gordon and Breach, New York, 1963).

54. R. Gonzalez and S. Payne, "User's Manual for NEMESIS and PLMODE," Applied Research Laboratories Technical Memorandum No. 80-6 (ARL-TM-80-6), Applied Research Laboratories, The University of Texas at Austin, 1 May 1980.

55. International Mathematical and Statistical Libraries, Inc., Houston, "The IMSL Library," Vol. 1, 6th edition, Reference Manual, 1979.

56. J. Turner, Buoyancy Effects in Fluids (Cambridge University Press, Cambridge, London, 1972) Chap. 8.

57. W. Horsthemke, lecture notes from an informal class on stochastic differential equations, given at The University of Texas at Austin, fall 1981 and spring 1982.

58. L. Arnold, op. cit. Ref. 9, Chap. 2, p. 29.

59. W. Horsthemke and R. Lefever, Noise-Induced Transitions (Springer-Verlag, Berlin, 1984).

60. I. Gradshteyn and I. Ryzhik, Table of Integrals, Series and Products (Academic Press, New York, 1965), p. 406, eq. 3.723.1.

61. I. Gradshteyn and I. Ryzhik, ibid., p. 407, eq. 3.723.5.

December 1984

DISTRIBUTION LIST FOR
ARL-TR-84-14
UNDER CONTRACT N00014-80-C-0490

Copy No.	
	Office of Naval Research 495 Summer Street Boston, MA 02210
1	Attn: L. Sternberg
	Commanding Officer Naval Ocean Research and Development Activity NSTL Station, MS 39529
2	Attn: E. D. Chaika (Code 530)
3	D. B. King (Code 321)
4	R. Wagstaff (Code 245)
5	K. Gilbert (Code 221)
6	A. Wenzel
	Chief of Naval Research Department of the Navy Arlington, VA 22217
7	Attn: M. McKisic (Code 425-OA)
8	R. Obrochta (Code 425 AR)
9	F. E. Saalfield (Code 400)
	Office of Naval Research Detachment Naval Ocean Research and Development Activity NSTL Station, MS 39529
10	Attn: G. Morris (Code 425GG)
	Director Naval Research Laboratory Department of the Navy Washington, D.C. 20375
11	Attn: O. Diachok (Code 5128)
12	R. Dicus (Code 5128)
13	R. Gragg
14	R. Pitre
15	D. Berman
16	J. Perkins
17	R. Baer
18	A. Tolstoy

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

Copy No.

Copy No.	
	Commanding Officer Naval Ocean Systems Center Department of the Navy San Diego, CA 92152
19	Attn: H. P. Bucker
	Officer in Charge New London Laboratory Naval Underwater Systems Center Department of the Navy New London, CT 06320
20	Attn: B. Cole
21	F. R. DiNapoli
22	P. Herstein
23	D. Lee
24	H. Weinberg
25	W. Roderick
26	B. Buehler
27	D. Wood
	Superintendent Naval Postgraduate School Monterey, CA 93940
28	Attn: Library
	Manager Anti-Submarine Warfare Systems Program Department of the Navy Washington, DC 20362
29	Attn: CDR K. E. Evans (PM-4)
	Naval Ocean Systems Center Department of the Navy San Diego, CA 92152
30	Attn: J. Neubert
	Woods Hole Oceanographic Institution 86-95 Water Street Woods Hole, MA 02543
31	Attn: R. Spindel
32	J. Lynch
33	G. Frisk

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

Copy No.

Copy No.	
	Science Applications, Inc. 1710 Goodridge Drive McLean, VA 22101
34	Attn: C. Spofford
35	J. Hanna
36	R. Greene
37	L. Dozier
38	M. Brill
	Applied Research Laboratory The Pennsylvania State University P. O. Box 30 State College, PA 16801
39	Attn: S. McDaniel
40	D. McCammon
	Marine Physical Laboratory of The Scripps Institution of Oceanography The University of California, San Diego San Diego, CA 92132
41	Attn: F. Fisher
	Defence Scientific Establishment HMNZ Dockyard Devonport, Auckland NEW ZEALAND
42	Attn: K. M. Guthrie
	School of Mechanical Engineering Georgia Institute of Technology Atlanta, GA 30332
43	Attn: A. D. Pierce
44	P. Rogers
	Department of Geology and Geophysics Geophysical and Polar Research Center Lewis G. Weeks Hall for Geological Sciences The University of Wisconsin, Madison 1215 W. Dayton Street Madison, WI 53706
45	Attn: C. S. Clay

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

<u>Copy No.</u>

	Courant Institute 251 Mercer Street New York, NY 10012	
46	Attn:	D. C. Stickler
47		G. Papanicolaou
	Hawaii Institute of Geophysics The University of Hawaii 2525 Correa Road Honolulu, HI 96822	
48	Attn:	L. N. Frazer
	Defence Research Establishment Pacific FMO Victoria, BC VOS 1BO CANADA	
49	Attn:	R. Chapman
	Defence Research Establishment Atlantic 9 Grove Street P. O. Box 1012 Dartmouth, NS CANADA	
50	Attn:	D. Chapman
51		I. Frazer
	Rosenteil School of Marine and Atmospheric Science The University of Miami 10 Rickenbacker Causeway Miami, FL 33149	
52	Attn:	H. DeFarrari
	Applied Physics Laboratory The Johns Hopkins University Johns Hopkins Road Laurel, MD 20810	
53	Attn:	A. Boyles
	Department of Ocean Engineering Massachusetts Institute of Technology Cambridge, MA 02139	
54	Attn:	I. Dyer

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

Copy No.

Copy No.	
	The University of Miami 10 Rickenbacker Causeway Miami, FL 33149
55	Attn: F. Tappert
56	M. Brown
	Department of Electrical Engineering Polytechnic Institute of New York Farmingdale, NY 11735
57	Attn: L. B. Felsen
	Centre Oceonologique de Bretagne Boite Postal 337 29273 Brest-Cedex FRANCE
58	Attn: Y. Desaubies
	Physics Department The University of Auckland Private Bag, Auckland NEW ZEALAND
59	Attn: A. C. Kibblewhite
60	C. T. Tindle
	Department of Physics The University of Texas at Austin Austin, TX 78712
61	Attn: W. Schieve
62	J. Turner
63	W. Horsthemke
	Department of Mechanical Engineering The University of Texas at Austin Austin, TX 78712
64	Attn: Ilene Busch-Vishniac
	Service de Chimie et Physique II Universite Libre de Bruxelles C.P. 231 Campus Plaine B-1050 Bruxelles BELGIUM
65	Attn: A. Garcia

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

Copy No.

Max Planck Institut für Meterologie
2 Hamburg 13
Bundesstrasse 55
FEDERAL REPUBLIC OF GERMANY (WEST GERMANY)
66 Attn: K. Hasselmann
67 C. Penland

Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24060
68 Attn: I. Beseiris
69 W. Kohler

CSNG-La Jolla Institute
8950 Villa La Jolla Drive, #2150
La Jolla, CA 92037
70 Attn: S. Flatte

Institute of Geophysics and Planetary Physics (A025)
Scripps Institute of Oceanography
University of California, San Diego
La Jolla, CA 92093
71 Attn: W. Munk

Department of Ocean Engineering
Florida Atlantic University
Boca Raton, FL 33431
72 Attn: D. Dunn
73 J. Cushchieri
74 Y. K. Lin
75 B. Davidson

Department of Physics
Florida Atlantic University
Boca Raton, FL 33431
76 Attn: S. Bruenn

Department of Mathematics
Florida Atlantic University
Boca Raton, FL 33431
77 Attn: F. Schroeck

Applied Physics Laboratory
University of Washington
Seattle, WA 98115
78 Attn: T. Ewart

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

<u>Copy No.</u>

	Catholic University Washington, DC 20064
79	Attn: J. McCoy
80	L. Fishman
	Applied Physics Laboratory The University of Washington 1013 NE 40 Street Seattle, WA 98105
81	Attn: M. Schulkin
	Institute for Advanced Study Princeton University Princeton, NJ 08540
82	Attn: R. Dashen
	Arete Associates P. O. Box 2951 Arlington, VA 22202
83	Attn: H. Freese
84	James A. TenCate 4210 Cherokee Grandville, MI 49418
85	Yves H. Berthelot, ARL:UT
86	David T. Blackstock, ARL:UT
87	Patrick L. Brockett, ARL:UT
88	Frederick D. Cotaras, ARL:UT
89	Karl C. Focke, ARL:UT
90	Terry L. Foreman, ARL:UT
91	Thomas A. Griffy, ARL:UT
92	Loyd Hampton, ARL:UT

Distribution list for ARL-TR-84-14 under Contract N00014-80-C-0490 (cont'd)

Copy No.	
93	Claude W. Horton, Sr., ARL:UT
94	David Knobles, ARL:UT
95	Robert A. Koch, ARL:UT
96	Jo B. Lindberg, ARL:UT
97	Stephen K. Mitchell, ARL:UT
98	David A. Nelson, ARL:UT
99	David W. Oakley, ARL:UT
100	Clark S. Penrod, ARL:UT
101	James Satterthwaite, ARL:UT
102	Paul J. Vidmar, ARL:UT
103	Library, ARL:UT
104 - 124	Reserve, ARL:UT

END

FILMED

3-85

DTIC